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ON CERTAIN CASES OF SIMPLE EXACT SOLUTIONS OF FLOW EQUATIONS IN A COMPRESSIBLE IMPERFECTLY VISCOUS FLUID WITH PARTICULAR CONDITIONS*

M. Z. Krzywoblocki

INTRODUCTION

In the most general case of flow in a compressible fluid the density, viscosity, specific heat at constant volume, specific heat at constant pressure and heat conductivity are functions of pressure and temperature. The solution of equations of motion together with equation of continuity and of energy present great difficulties. The physical conditions superimpose certain boundary values on velocity, density, pressure and temperature. Because the energy equation expresses only in an analytical way the law of conservation of energy with all its possible forms like intrinsic, kinetic, etc. one more restrictive condition must be superimposed on the whole system. This usually contains the thermodynamic relationship between heat added, intrinsic energy and external work. It may refer also to the value of heat content per unit mass. This picture shows that at the present time the difficulties in the solution of equations are probably insurmountable. But it is possible to select cases satisfying only a part of the restrictive conditions, which although uninterpretable in all details from a physical standpoint, are solvable from a mathematical point of view. In the present paper three such cases are solved under the conditions that the coefficients of viscosity and thermal conduction are constant, that only one boundary condition is superimposed, namely the one concerning the velocity at infinity and that all other restrictive conditions are neglected. The cases are: two-dimensional source, circular vortex, and spiral vortex. The solution was obtained in the form of exponential functions and the worked-out examples represent the cases of exact formal solutions of equations under accepted conditions. This last is the main aim of the paper. The obtained results show that from the four parameters two *i.e.* velocity and density are always interpretable from a physical standpoint, but two others *i.e.* temperature and pressure are, in some cases, uninterpretable from a physical standpoint. In certain cases they came out to be negative in the whole plane.

The case of a source was reduced to adiabatic conditions. Also the case of a three-dimensional source in adiabatic conditions was solved

* Presented to the American Mathematical Society, at New York February 22, 1947

by the used method.* The author did not succeed in solving other cases by the applied method. The considered cases are the simplest cases in the theoretical fluid dynamics. The obtained results seem to show that the number of cases with exact solutions in the theory of compressible viscous fluid, physically interpretable in all the items, will be probably very limited.**

1. *Vectorial form of equations.*

The equation of motion of a compressible viscous fluid can be represented in the following vectorial form:

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + \text{grad}\,(\tfrac{1}{2}\vec{V}^2) - (\vec{V}\times\vec{\omega})\right] = \tag{1}$$

$$\rho\vec{F} - \text{grad}\,p + \tfrac{1}{3}\mu\,\text{grad div}\,\vec{V} + \mu(\text{grad div}\,\vec{V} - \text{curl}\,\vec{\omega}).^{***}$$

The equation of continuity:

$$\frac{\partial\rho}{\partial t} + \vec{V}\cdot\text{grad}\,\rho + \rho\,\text{div}\,\vec{V} = 0. \tag{2}$$

The equation of energy:

$$\rho Jc_v\left[\frac{\partial T}{\partial t} + \vec{V}\cdot\text{grad}\,T\right] + p\,\text{div}\,\vec{V} = Jk\,\text{div grad}\,T + \Phi\,, \tag{3}$$

$$\Phi = \mu\Big[2(\text{grad}\,u\cdot\text{grad}\,u + \text{grad}\,v\cdot\text{grad}\,v + \text{grad}\,w\cdot\text{grad}\,w) - \tag{3a}$$

$$- \text{curl}\,\vec{V}\cdot\text{curl}\,\vec{V} - \tfrac{2}{3}(\text{div}\,\vec{V})^2\Big].$$

The equation of state:

$$p = \bar{R}\rho T. \tag{4}$$

The used symbols denote:

- F – extraneous forces,
- p – pressure,
- ρ – mass density,
- μ – coefficient of viscosity,
- J – mechanical equivalent of heat,
- k – thermal conductivity,
- c_v – specific heat at constant volume
- T – absolute temperature
- Φ – dissipation function,
- $\vec{\omega}$ – curl $\vec{V}$.

* The three worked-out examples in two-dimensional flow were solved for adiabatic conditions in case of an ideal fluid ($\mu=k=0$) by F. Ringleb. Z.A.M.M., V.20, 1940, p.13. All the restrictive conditions were taken into account. The obtained solution is in a form of hypergeometric series. See also the solution of Taylor: References in H.Bateman. Compressible Fluids. Bulletin of the National Research Council, No.84, Feb., 1932, The National Research Council, Washington D.C.

**The number of cases with exact solutions in the theory of incompressible viscous fluid is also very limited (flow between two non-parallel walls, flow due to a rotating disc, etc.).

***Let it be assumed that the Z-axis coincides with the vertical axis. In this case: $\vec{F} = -\text{grad}\,\Omega \pm \vec{G}$, $-\text{grad}\,\Omega = \vec{i}X + \vec{j}Y + \vec{k}Z$, $\vec{G} = \vec{k}g(\frac{\rho - \rho_1}{\rho})$.

For perfect gases the specific heat at constant volume, c_v, and the specific heat at constant pressure, c_p, are constant. In all the equations below it is assumed that the coefficients c_v, c_p, μ and k are constant.

2. *Cylindrical polar coordinates.*

In cylindrical polar coordinates equations of motions are:

$$\rho u_t + \rho(uu_r + vr^{-1}u_\phi + wu_z - v^2r^{-1}) = \rho F_r - p_r + \mu\Big\{\frac{4}{3}[u_{rr} + r^{-1}u_r] + r^{-2}u_{\phi\phi} + u_{zz} - \frac{1}{3}[4ur^{-2} + 7r^{-2}v_\phi - r^{-1}v_{r\phi} - w_{rz}]\Big\}, \tag{5a}$$

$$\rho v_t + \rho(uv_r + vr^{-1}v_\phi + wv_z + uvr^{-1}) = \rho F_\phi - r^{-1}p_\phi + \mu\Big\{\frac{1}{3}[r^{-1}u_{r\phi} + 7r^{-2}u_\phi] + v_{rr} + r^{-1}v_r - vr^{-2} + v_{zz} + \frac{1}{3}[4r^{-2}v_{\phi\phi} + r^{-1}w_{\phi z}]\Big\}, \tag{5b}$$

$$\rho w_t + \rho(uw_r + vr^{-1}w_\phi + ww_z) = \rho F_z - p_z + \mu\Big\{\frac{1}{3}\, u_{rz} + r^{-1}u_z + v_{\phi z}] + w_{rr} + r^{-1}w_r + r^{-2}w_{\phi\phi} + \frac{4}{3}w_{zz}\Big\}. \tag{5c}$$

Continuity equation:

$$\rho_t + r^{-1}[(ru\rho)_r + (v\rho)_\phi] + (w\rho)_z = 0. \tag{6}$$

Energy equation

$$\rho Jc_v[T_t + uT_r + vr^{-1}T_\phi + wT_z] + p[u_r + r^{-1}(u + v_\phi) + w_z] = Jk(T_{rr} + r^{-1}T_r + r^{-2}T_{\phi\phi} + T_{zz}) + \Phi, \tag{7}$$

$$\Phi = \mu\Big\{2[u_r^2 + v_r^2 + r^{-2}(v_\phi^2 - (vr)_r^2) + w_z^2] + (r^{-1}w_\phi + v_z)^2 + (u_z + w_r)^2 + r^{-2}[(vr)_r + u_\phi]^2 - \frac{2}{3}[u_r + r^{-1}(u + v_\phi) + w_z]^2\Big\}. \tag{7a}$$

Eq. (4) remains unchanged. Subscripts denote partial differentiations except in case of F where they denote components. The vertical force component F_z is equal to: $F_z = Z \pm g(\frac{\rho - \rho_1}{\rho})$.

3. *Spherical polar coordinates.*

With spherical polar coordinates R, ν, ϕ, the following corresponding equations are obtained:

Equations of motion:

$$\rho[u_t + uu_R + R^{-1}vu_\nu + (R\sin\nu)^{-1}wu_\phi - R^{-1}(v^2 + w^2)] = \rho F_R + g(\rho - \rho_1)\cos\nu - p_R + \mu\Big\{\frac{8}{3}R^{-1}(u_R - R^{-1}u) + \frac{4}{3}u_{RR} + 2R^{-2}[(\tan\nu)^{-1}u_\nu + u_{\nu\nu} + (\sin\nu)^{-2}u_{\phi\phi}] + (3R)^{-1}[(\tan\nu)^{-1}v_R + v_{R\nu}] - \frac{7}{3}R^{-2}[(\tan\nu)^{-1}v + v_\nu] - (3R\sin\nu)^{-1}(7R^{-1}w_\phi - w_{R\phi})\Big\}, \tag{8a}$$

$$\rho[v_t + R^{-1}uv + uv_R + R^{-1}vv_\nu + (R\ \sin\nu)^{-1}wv_\phi - (R\ \tan\nu)^{-1}w^2] =$$
$$\rho F_\nu + g(\rho - \rho_1)\ \sin\nu - R^{-1}p_\nu + R^{-1}\mu\left\{\frac{4}{3}(2R^{-1}u_\nu + u_{R\nu}) + R(\sin\nu)u_{R\nu} + \right.$$
$$\frac{4}{3}R^{-1}[(\tan\nu)^{-1}v_\nu - (\sin\nu)^{-2}v + v_{\nu\nu}] + R\ \sin\nu\ (2v_R + Rv_{RR}) +$$
$$\left. R^{-1}(\sin\nu)^{-2}v_{\phi\phi} - (3R\ \sin\nu)^{-1}[7(\tan\nu)^{-1}w_\phi - w_{\nu\phi}]\right\}, \qquad (8b)$$

$$\rho[w_t + R^{-1}uw + uw_R + R^{-1}vw_\nu + (R\ \tan\nu)^{-1}vw + (R\ \sin\nu)^{-1}ww_\phi] =$$
$$\rho F_\phi - (R\ \sin\nu)^{-1}p_\phi + (R\ \sin\nu)^{-1}\mu\left\{\frac{4}{3}(2R^{-1}u_\phi + u_{R\phi}) + \right.$$
$$(3R)^{-1}[7(\tan\nu)^{-1}v_\phi + v_{\nu\phi}] + \sin\nu\ (2w_R + Rw_{RR} + R^{-1}w_{\nu\nu}) +$$
$$\left. R^{-1}[w_\nu\ \cos\nu - w(\sin\nu)^{-1} + \frac{4}{3}(\sin\nu)^{-1}w_{\phi\phi}]\right\}. \qquad (8c)$$

Continuity equation:

$$\rho_t + R^{-1}\left\{R^{-1}(R^2u\rho)_R + (\sin\nu)^{-1}[(v\rho\ \sin\nu)_\nu + (w\rho)_\phi]\right\} = 0. \qquad (9)$$

Energy equation

$$\rho Jc_v\left\{T_t + uT_R + R^{-1}[vT_\nu + (\sin\nu)^{-1}wT_\phi]\right\} +$$
$$p[2R^{-1}u + u_R + (R\ \sin\nu)^{-1}(v\ \cos\nu + w_\phi) + R^{-1}v_\nu] =$$
$$Jk\left\{2R^{-1}T_R + T_{RR} + R^{-2}[(\tan\nu)^{-1}T_\nu + T_{\nu\nu}] + R^{-2}(\sin\nu)^{-2}T_{\phi\phi}\right\} + \Phi, \qquad (10)$$

$$\Phi = \mu\left\{2[u_R^2 + v_R^2 + w_R^2 + R^{-2}(u_\nu^2 + v_\nu^2 + w_\nu^2) + (R\ \sin\nu)^{-2}(u_\phi^2 + v_\phi^2 + w_\phi^2)] - \right.$$
$$[(R\ \sin\nu)^{-1}u_\phi - R^{-1}w - w_R]^2 - (R^{-1}v + v_R - R^{-1}u_\nu)^2 - [(R\ \tan\nu)^{-1}w + R^{-1}w_\nu -$$
$$\left.(R\ \sin\nu)^{-1}v_\phi]^2 - \frac{2}{3}[R^{-1}(2u + v_\nu) + u_R + (R\ \sin\nu)^{-1}(v\ \cos\nu + w_\phi)]^2\right\}. \qquad (10a)$$

4. *Two-dimensional source in steady flow.*

For a source (or a sink) at the origin in a steady two-dimensional flow with no extraneous forces equations (5), (6), (7) with $v = 0$ give:

$$\rho uu_r = -p_r + \mu[\frac{4}{3}(u_{rr} + r^{-1}u_r - r^{-2}u) + r^{-2}u_{\phi\phi}], \qquad (11a)$$

$$r^{-1}p_\phi - \mu(3r)^{-1}(7r^{-1}u_\phi + u_{r\phi}) = 0, \qquad (11b)$$

$$r^{-1}(ur\rho)_r = 0, \qquad (11c)$$

$$\rho Jc_v uT_r + p(r^{-1}u + u_r) = Jk(r^{-1}T_r + T_{rr} + r^{-2}T_{\phi\phi}) + \Phi, \qquad (11d)$$

$$\Phi = \mu[2u_r^2 + r^{-2}u_\phi^2 - \frac{2}{3}(r^{-1}u + u_r)^2]. \qquad (11e)$$

Because of symmetry conditions there is: $u_\phi = p_\phi = T_\phi = 0$. Hence eq. (11b) drops out. All the dependent variables will depend only on r. Assume a circle around the origin of an arbitrary radius r_1. The following equations will be taken into account: *

$$\rho u u' + p' - \frac{4}{3}\mu(r^{-1}u' + u'' - r^{-2}u) = 0, \tag{12a}$$

$$r u \rho = r_1 u_1 \rho_1 = \text{const.}, \tag{12b}$$

$$\rho J c_v u T' + p(r^{-1}u + u') - Jk(r^{-1}T' + T'') - \Phi = 0. \tag{12c}$$

$$\Phi = \frac{2}{3}\mu[2u'^2 - 2r^{-1}uu' - r^{-2}u^2], \tag{12d}$$

$$p = \overline{R}\rho T. \tag{12e}$$

5.1. *Differential equations.*

Elimination of p and ρ from eqs. (12a) and (12c) by use of (12b) and (12e) gives in the result the following two equations in dimensionless form:

$$3\left(\frac{r_1 u_1}{\nu_1 u}\right)[(r - \overline{R}u^{-2}rT)u' - \overline{R}u^{-1}(T - rT')] - 4[u^{-1}r(u' + ru'') - 1] = 0, \tag{13a}$$

$$3\left(\frac{r_1 u_1}{\nu_1 u^2}\right)[\overline{R}T(u^{-1}ru' + 1) + Jc_v rT'] - 3J\frac{k}{\mu}\frac{r}{u^2}(T' + rT'') + 4u^{-1}ru'(1 - u^{-1}ru') + 2 = 0, \tag{13b}$$

where the symbol $\nu_1 = \frac{\mu}{\rho_1}$ denotes the coefficient of kinematic viscosity. As the known values assume the strength of the source $K = r_1 u_1$ and both factors contained in K and the density ρ_1.

By introducing the following transformations:

$$U_{(\mathfrak{z})} = \left(\frac{u^2 r_1}{\nu_1 u_1}\right), \quad \theta_{(\mathfrak{z})} = \overline{R}T\left(\frac{r_1}{\nu_1}\right)^2, \quad \mathfrak{z} = \log\frac{r}{r_1},$$

$$\frac{dU}{d\mathfrak{z}} = U', \quad \frac{d\theta}{d\mathfrak{z}} = \theta', \quad \text{etc.},$$

$$\frac{du}{dr} = \frac{1}{2}\left(\frac{\nu_1 u_1}{r_1}\right)^{1/2} r^{-1}U^{-1/2}U', \quad \text{etc.},$$

$$\frac{dT}{dr} = \left(\frac{\nu_1}{r_1}\right)^2 (\overline{R}r)^{-1}\theta', \quad \frac{d^2T}{dr^2} = \left(\frac{\nu_1}{r_1 r}\right)^2 \overline{R}^{-1}(\theta'' - \theta'), \tag{14}$$

eqs. (13) may be easily transformed into the equations:

* u', p', T', means differentiation with respect to r.

$$U[8U + 3\alpha U' - 4U'' - 6(\theta - \theta')] + U'(2U' - 3\theta) = 0, \qquad (15a)$$

$$2U[2(U + U') + 3(\theta + \delta\theta' - \beta\theta'')] - U'(2U' - 3\theta) = 0, \qquad (15b)$$

where the symbols used denote:

$$\alpha = r_1 u_1 v_1^{-1},\ \beta = Jk(r_1 u_1 \rho \bar{R})^{-1} = Jk(\alpha\mu\bar{R})^{-1},\ \delta = Jc_v \bar{R}^{-1}. \qquad (16)$$

Addition of eqs. (15) gives

$$12U + (3\alpha + 4)U' - 4U'' + 6[(1 + \delta)\theta' - \beta\theta''] = 0. \qquad (17)$$

5.2 *Boundary conditions and solution of differential equations.*

The only boundary conditions which are superimposed on the whole system are those referring to the velocity, namely:

$$u = u_1 \text{ and } U = U_0 = \frac{u_1 r_1}{v_1} \text{ for } r = r_1 \text{ or } \mathfrak{z} = 0,$$

$$u \to 0 \text{ and } U \to 0 \quad \text{for } r \to \infty \text{ or } \mathfrak{z} \to \infty . \qquad (18)$$

Putting

$$U = a_0 + \frac{a_1}{c^2} e^{-c\mathfrak{z}},\quad \theta = \frac{b_1}{c^2} e^{-c\mathfrak{z}}, \qquad (19)$$

and imposing the boundary conditions, the result is:*

$$a_0 = 0,\ a_1 = c^2 u_1 r_1 v_1^{-1}. \qquad (20)$$

The unknown values b_1 and c may be determined from eqs. (15) which take the form:

$$a_1(8 - 3\alpha c - 2c^2) - 3b_1(2 + c) = 0, \qquad (21)$$

$$2a_1[2(1 - c) - c^2] + 3b_1[2(1 - \delta c) - c(1 + 2\beta c)] = 0, \qquad (21a)$$

whereas eq. (17) becomes:

$$(4a_1 + 6\beta b_1)c^2 + [(3\alpha + 4)a_1 + 6(\delta + 1)b_1]c - 12a_1 = 0, \qquad (22)$$

or

$$Ac^2 + Bc - C = 0 \qquad (22a)$$

The unknown c calculated from eq. (22a) has the form:

$$c = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} . \qquad (23)$$

Using the positive value of the square root it is always possible to find a positive value of c. The coefficients α, β, δ, a_1, are always

* It may be an interesting problem to prove that this is a unique solution of Eqs. (15).

positive, whereas b_1 may have any real value. In the case where $A > 0$, the absolute value of the square root is greater than the absolute value of B and $c > 0$. In the case where $A < 0$, the absolute value of the square root is smaller than B but also the denominator is negative and and in the result $c > 0$. Substitution of a value for c into one of equations (21) gives the value of b_1. One may show that the coefficient b_1 takes at least one real value. Equation (21a) after substituting the value for c becomes:

$$f(b_1) = (c_1^2 - c_5^2c_8)b_1^6 + (2c_1c_2 - 2c_5c_6c_8 - c_5^2c_9)b_1^5 +$$
$$[2c_1c_3 + c_2^2 - c_6^2c_8 - 2c_5(c_7c_8 + c_6c_9) - c_5^2c_{10}]b_1^4 +$$
$$[2(c_1c_4 + c_2c_3 - c_6c_7c_8) - c_6^2c_9 - 2c_5(c_7c_9 + c_6c_{10})]b_1^3 +$$
$$[2\ c_2c_4 + c_3^2 - c_7^2c_8 - 2c_6c_7c_9 - c_6^2c_{10} - 2c_5c_7c_{10}]b_1^2$$
$$(2c_3c_4 - c_7^2c_9 - 2c_6c_7c_{10})b_1 + (c_4^2 - c_7^2c_{10}) = 0, \qquad (24)$$

where the coefficients used have the following values:

$$\begin{aligned}
c_1 &= 216\beta(1 - 4\beta + \delta),\\
c_2 &= 12a_1(27\alpha\beta + 18\alpha\beta\delta - 84\beta + 95\beta^2 - 12\delta^2),\\
c_3 &= 2a_1^2[36\alpha(1 + 2\beta) + 54\alpha^2\beta + 481\beta - 96\delta - 240],\\
c_4 &= 2a_1^3(31 + 18\alpha^2),\\
c_5 &= -36\beta,\\
c_6 &= 12a_1(2\delta - 3\alpha\beta),\\
c_7 &= 4a_1^2(4 - 3\alpha),\\
c_8 &= 36(1 + \delta)^2,\\
c_9 &= 12a_1[4(1 + \delta) + 3\alpha(1 + \delta) + 48\beta],\\
c_{10} &= a_1^2(9\alpha^2 + 24\alpha + 208).
\end{aligned} \qquad (25)$$

The value of the function for $b_1 = 0$ depends upon the last term in (24) which after all the necessary transformations may be presented in the form:

$$(c_4^2 - c_7^2c_{10}) = 4a_1^6(864\alpha^3 - 7812\alpha^2 + 22272\alpha - 13887). \qquad (26)$$

The expression inside the bracket is positive for $0.9 < \alpha$, negative for $0 < \alpha < 0.8$. Since it is not probable that the small value of $\alpha < 0.8$ may be taken into account, it is assumed that $f(b_1)_{b_1=0} > 0$. The value of the function $f(b_1)_{b_1\to\infty}$ for $b_1 \to \infty$ depends upon the first coefficient of eq. (24) which takes the form:

$$(c_1^2 - c_5^2 c_8) = 373248\beta^2(2\beta - \delta - 1). \tag{27}$$

This is negative unless β is great or α small which was excluded.* This means that there exists at least one real root between $f(b_1)_{b_1=0}$ and $f(b_1)_{b_1\to\infty}$.

The final results are:

$$u = u_1\left(\frac{r_1}{r}\right)^{\frac{c}{2}}, \quad \rho = \rho_1\left(\frac{r_1}{r}\right)^{1-\frac{c}{2}}, \quad T = \frac{b_1}{\overline{R}}\left(\frac{v_1}{c}\right)^2\frac{r_1^{c-2}}{r^c}, \quad p = b_1\frac{1}{c^2}\rho_1 v_1\frac{r_1^{\frac{c}{2}-1}}{r^{\frac{c}{2}+1}}. \tag{28}$$

One may easily derive the formula for the polytropic relation in the case considered:

$$p = C\rho^{\psi}, \text{ or } p = C_1 f(r)\rho^{\gamma}, \tag{28a}$$

$$C = b_1 c^{\frac{4}{c-2}}\left(\frac{u_1}{a_1}\right)^{\frac{c}{c-2}}\left(\frac{r_1}{\mu}\right)^{\frac{4-c}{c-2}}\rho_1^{\frac{4+c}{c-2}}, \quad \psi = \frac{2+c}{2-c}, \tag{28b}$$

$$C_1 = Cr_1^{\frac{(2-c)(\psi-\gamma)}{2}}\rho_1^{(\psi-\gamma)}, \quad f(r) = r^{\left(\frac{c-2}{2}\right)(\psi-\gamma)}, \quad \gamma = \frac{c_p}{c_v}.$$

5.3 *Particular cases*

Some particular cases may be taken into account, although it may be difficult to interpret them from a physical standpoint.

(a) $k = 0$. In this case $\beta = 0$ and the general procedure remains unchanged.

(b) $\mu = 0$. Applying the substitutions:

$$U = \left(\frac{u}{u_1}\right)^2, \quad \theta = \overline{R}Tu_1^{-2}, \quad \mathfrak{z} = \log\left(\frac{r}{r_1}\right), \quad \beta = Jk(\overline{R}r_1u_1\rho_1)^{-1}, \tag{29}$$

gives the following results:

$$U[U' - 2(\theta - \theta')] - U'\theta = 0, \tag{29a}$$

$$2U[\theta + 2(\delta\theta' - \beta\theta'')] + U'\theta = 0. \tag{29b}$$

* Both experiments and more accurate calculations give values around 2 for $k/(\mu c_v)$ (See: Modern Developments in Fluid Dynamics, Edited by S. Goldstein. Oxford, Clarendon Press, 1938, p. 10.). Hence 2β is equal approximately to $(4/\alpha)\delta$. For most gases $\delta \approx 2.5$ to 3. Thus with $\delta = 3$, formula (27) gives positive values for $\alpha < 3$.

Keeping eq. (19) and preserving the same boundary conditions eqs. (29a) and (29b) give:

$$a_1 c + (2 + c)b_1 = 0, \tag{30a}$$

$$2(\beta c^2 - 1) + (2\delta + 1)c = 0. \tag{30b}$$

The second eq. gives the value of c which is always a positive one. From the first eq. the value of b_1 is:

$$2(\beta - \delta - 1)b_1^2 - a_1(2\delta + 3)b_1 - a_1^2 = 0. \tag{30c}$$

It is easily shown that the value of b_1 is always a real one. The final results are:

$$a_1 = c^2, \; u = u_1\left(\frac{r_1}{r}\right)^{\frac{c}{2}}, \; \rho = \rho_1\left(\frac{r_1}{r}\right)^{1-\frac{c}{2}},$$

$$T = \frac{b_1}{\bar{R}}\left(\frac{u_1}{c}\right)^2\left(\frac{r_1}{r}\right)^c, \; p = \rho_1 b_1\left(\frac{u_1}{c}\right)^2\left(\frac{r_1}{r}\right)^{\frac{c}{2}+1}. \tag{31}$$

(c) $k = \mu = 0$. In this case $\beta = 0$. It is the case where purely adiabatic conditions for ideal fluids exist:

$$c = 2\frac{1}{(2\delta + 1)}, \qquad b_1 = -\frac{a_1 c}{(2+c)}. \tag{32}$$

Using eq. (4) with (12b) and (29) after simple transformations the result is:

$$p = C\rho^{\frac{2+c}{2-c}}, \tag{33}$$

$$C = u_1^2\frac{b_1}{c^2}\rho_1^{\frac{2c}{c-2}}$$

After application of (32) with $\bar{R} = J(c_p - c_v)$:

$$\frac{2 + c}{(2 - c)} = \frac{c_p}{c_v} \tag{34}$$

which is correct.

6. *Circular vortex in steady flow.*

Let there be a single cylindrical vortex tube, whose cross section is a circle of radius r_1, surrounded by unbounded liquid. Let the vorticity be assumed to have the constant value ω over the area of the section. Inside the vortex the motion is rotational. The origin will be taken at the center of the circle. In the vortex motion the radial velocity u will be equal to zero and all other dependent variables will be functions of r only. Hence under the assumption that the motion is steady and no extraneous forces are acting, eqs. (5), (6) and (7) take the form:

$$\rho v^2 r^{-1} - p' = 0, \tag{35a}$$

$$v'' + r^{-1}(v' - r^{-1}v) = 0, \tag{35b}$$

$$r^{-1}(\rho v)_\phi = 0, \tag{36}$$

$$Jk(T'' + r^{-1}T') + \mu[v'^2 - 2r^{-1}vv' - r^{-2}v^2] = 0. \tag{37}$$

The equation of state is of course valid. The equation of continuity shows that in fact if v is a function of r only, ρ must be also a function of r only. The use of the following notations:

$$V(\mathfrak{z}) = \left(\frac{v}{v_1}\right)^2 = \frac{a_1}{c^2}\exp(-c\mathfrak{z}),\ \theta(\mathfrak{z}) = \bar{R}Tv_1^{-2} = \frac{b_1}{c^2}\exp(-c\mathfrak{z}),$$

$$\Pi(\mathfrak{z}) = \rho,\ \epsilon = \frac{Jk}{\bar{R}\mu} \tag{38}$$

gives the results:

$$V - \Pi^{-1}\Pi'\theta - \theta' = 0,\ \text{or}\ \Pi^{-1}\Pi' = \theta^{-1}(V - \theta'), \tag{38a}$$

$$2V(2V - V'') + V'^2 = 0, \tag{38b}$$

$$4V(V + V' - \epsilon\theta'') - V'^2 = 0. \tag{38c}$$

The second equation gives $c = 2$, the third $b_1 = -\frac{\epsilon^{-1}}{2}$ and from the first:

$$\Pi = Cr^{-\lambda},\ \lambda = 2(\epsilon - 1),\ C = r_1^{\lambda}\rho_1. \tag{39a}$$

The other values are:

$$v = v_1 r_1 r^{-1},\ T = -\mu(2Jk)^{-1}(v_1 r_1)^2 r^{-2}. \tag{39b}$$

As can be seen the knowledge of the value of the strength of the vortex $K = \frac{1}{2}r_1^2\omega$ or of the circulation along a circle enclosing the cylinder $\Gamma = 2r\pi v$ and of the density ρ_1 is sufficient to find the exact values v, ρ, T, p.

The relation between pressure and density may be given by formulae:

$$p = C\rho^{\psi}\ , \text{ or } \ p = C_1 f(r)\rho^{\gamma}\ ,$$

$$C = b_1 c^{-2} v_1^2 \rho_1^{-\frac{c}{\lambda}}\ , \ \psi = (\lambda + c)c^{-1}\ ,$$

$$C_1 = C r_1^{\lambda(1-\gamma)+c}\ \rho_1^{1-\gamma+\frac{c}{\lambda}}\ , \ f(r) = r^{\lambda(\gamma-1)-c} \tag{39c}$$

6.1 *Particular cases.*

(*a*) $\mu = 0$. In this case eq. (38b) and values for c and v remain unchanged. From (38c) it can be seen that $\theta'' = 0$ or:

$$\theta = k_0(1 \pm \mathfrak{z}). \tag{40a}$$

Assuming that the initial value of T_1 for $r = r_1$, is given, results in:

$$k_0 = \theta_0 = \bar{R}T_1 v_1^{-2}\ . \tag{40b}$$

From (38a):

$$\Pi = C \exp\ (\textstyle\int Q\ dr), \tag{40c}$$

$$Q = \frac{(q - r^2)}{r^3 f(r)}\ , \ q = (v_1 r_1)^2 (\bar{R}T_1)^{-1}\ , \ f(r) = (1 \pm \log\frac{r}{r_1}). \tag{40d}$$

The value of C may be found from the initial conditions ($r=r_1$, $\rho=\rho_1$).

(*b*) $k = 0$. Eqs. (38b) and (38c) give different values for c. The system is not solvable.

(*c*) $\mu = k = 0$. In this case the energy equation drops out. The remaining set of equations cannot be solved without additional assumptions.

7. *Spiral vortex in steady flow.*

In this case both velocity components u and v and all other dependent variables depend only on r. Using the notations:

$$U(\mathfrak{z}) = \left[\frac{u}{u_1}\right]^2,\ V(\mathfrak{z}) = \left[\frac{v}{u_1}\right]^2,\ \theta(\mathfrak{z}) = \bar{R}Tu_1^{-2},\ \epsilon = Jk(\mu\bar{R})^{-1}, \tag{41}$$

and eq. (12b), eqs. (5a), (5b), and (7) may easily be transformed into the expressions:

$$U[4U + 6\alpha U' - 2U'' - 12\alpha(V - \theta + \theta')] + U'(U + 6\alpha\theta) = 0, \tag{41a}$$

$$2V[2V(1 + \alpha) + \alpha V' - V''] + V'^2 = 0, \tag{41b}$$

$$4UV[2(U+U')+3(V+V')]-3UV'^2+12UV[\alpha(\theta+\delta\theta')-\epsilon\theta'']+U'V(6\alpha\theta-U') = 0. \tag{41c}$$

Applying (19) with $V = d_1 c^{-2}\exp(-c\mathfrak{z})$ and the boundary conditions, the value of $a_1 = c^2$ is obtained, and the values of b_1, d_1, c, can be calculated from three equations:

$$c^4 + 6ac^3 - 4c^2 - 6ab_1 c - 12a(b_1 - d_1) = 0, \tag{42a}$$

$$(1 - 2a)c^2 - 2ac + 4(1 + a) = 0, \tag{42b}$$

$$c^4 + 8c^3 + (12\epsilon b_1 + 3d_1 - 8)c^2 + 6[(2\delta+1)ab_1 + 2d_1]c + 12(ab_1 + d_1) = 0. \tag{42c}$$

From (42b) it follows that for $a > \frac{1}{2}$ always $c > 0$.
The relation between pressure and density may be expressed by formulae:

$$p = C\rho^{\psi}, \text{ or } p = C_1 f(r)\rho^{\gamma}, \tag{43}$$

$$C = a_1^{\frac{c}{2-c}} b_1 c^{\frac{4}{c-2}} u_1 \rho_1^{\frac{2c}{c-2}}, \quad \psi = \frac{2+c}{2-c},$$

$$C_1 = Ca_1^{\frac{1}{2}(\gamma-\psi)} c^{(\psi-\gamma)} r_1^{\frac{1}{2}(2-c)(\psi-\gamma)} \rho_1^{(\psi-\gamma)}, \quad f(r) = r^{\frac{1}{2}(c-2)(\psi-\gamma)}$$

7.1 *Particular cases.*

(a) $k = 0$. In this case $\epsilon = 0$ and the procedure remains unchanged. No other particular cases can be solved by the used method.

8. *Three-dimensional source in steady flow.*

It may be of interest to know whether the method explained can be applied to other cases. In the case of a source in space the method explained above gives results only for $\mu = k = 0$.* The continuity eq. gives the condition: $u\rho R^2 = \text{const}$. Using this condition and (29) the following equations can be obtained from (8a) and (10):

$$U(U' - 4\theta + 2\theta') - U'\theta = 0, \tag{44a}$$

$$U(4\theta + 2\delta\theta') + U'\theta = 0. \tag{44b}$$

Application of (19) gives the result:

$$c = 4\,\frac{1}{(2\delta + 1)}, \quad b_1 = -\frac{c^3}{(4\quad c)}. \tag{45}$$

After some easy transformations it is possible to represent the equation of state in the form:

$$p = C\rho^{a},$$

* In order for the method to work, the dimensions of μ and $u\rho R^2$ must be the same. Besides the restrictive condition $\mu = k = 0$, also the term $g(\rho - \rho_1)\cos\nu$ must be neglected.

$$\alpha = \frac{(4 + c)}{(4 - c)}, \quad C = u_1^2 \frac{b_1}{c^2} \rho_1^{\frac{2c}{(c-4)}} \tag{46}$$

Similarly as in (34) it can be shown that $(4 + c):(4 - c) = \frac{c_p}{c_v}$

9. *Thermodynamic conditions*

As is seen, in some cases, given above, the heat content per unit mass, expressed by the formulae: $i = c_p T = \frac{k}{(k-1)} \frac{p}{\rho} = \frac{1}{(k-1)} a^2$ (a = velocity of sound) is a negative one. In the adiabatic cases the total energy per unit mass given by the expression $\frac{1}{2} \frac{1}{J} v^2 + i = i_0$ = const., is equal to zero.* This is a consequence of the assumed boundary value of the velocity. Because of that, the values of the density, pressure and temperature at infinity also tend to zero. Hence for a finite value of velocity the heat content comes out to be a negative one. The superimposing of a finite value of that energy in those cases will not permit applying the simple method which was used to obtain an exact solution of the equations. In the examples of an adiabatic flow in a source, worked out by other authors (see Ringleb's paper, loc. cit.), the total energy per unit mass had *a priori* a prescribed finite positive value. Consequently the value of density (and pressure) at infinity was a finite one.** Near the origin of the source, where the velocity was very great, the density and consequently the pressure tended to zero. In the present paper the density, pressure and temperature at infinity tend to zero and consequently in the whole plane and near the source origin the pressure and temperaeure have negative values. The density has everywhere on and outside the limiting circle a positive value. The maximum velocity obtainable in Ringleb's calculatîons is a finite one (equal to $2.236a_0$, it corresponds to $\rho_0 = 0$), but the motion ceases to be adiabatic beginning at a certain point before this maximum velocity is reached. In the present paper in the cases of adiabatic flow there is no limit superimposed on the value of the velocity, depending upon the abstractive negative values of the heat content and temperature. The motion is adiabatic in the whole plane outside the limiting circle. Thus the examples of exact formal solutions of equations of flow of a compressible viscous (to a certain rate) fluid which were worked out represent particular cases with solutions giving abstractive negative pressure and temperature.***

* Calculation of this energy in the two cases worked out above for adiabatic flow gives the value of the total energy equal to zero.

** There are signigicant differences between assumptions in the present and previous papers: the assumption of a finite value of i_0 causes that at a certain point on the radius near the origin, where the velocity reaches the value equal to $2.236a_0$, the density tends to zero. This item seems to be uninterpretable from the standpoint of a finite source strength and continuity. The assumption of a finite source strength causes that at infinity the density tends to zero, and as a consequence of that, i_0 is equal to zero.

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University of Illinois

ALGEBRA OF NEURAL NETS

Duane Studley

Abstract: A paper showing the application of modern mathematics in modern biophysics. As example Boolean algebra is shown as applied to neural nets.

Boolean algebra is a new algebra that was initiated by George Boole in 1854 with publication of his *Laws of Thought*. In the modern notation it goes like this: let a closed curve A denote a set of objects while another closed curve *B* denotes another set of objects. If the two sets have objects in common the two curves overlap as in the figure.

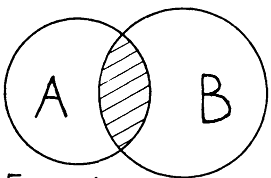

Figure 1

Instead of a sum we think of a union of sets *A* and *B* meaning the set of all objects in *A* or *B* or both. Boole originally excluded the both part of this concept thus obtaining what is now called modulo two addition. His idea survives in analysis situs, in formal logic as exclusive disjunction and other places where modulo two addition is used.

The overlap or common part is called the intersection of the sets *A* and *B*. And we write $A \cup B$ for the union and $A \cap B$ for the intersection of the sets *A* and *B*. Intersection then is the equivalent of multiplication. Then we have the notion of inclusion. When I write $A \cap B \subset A$ I mean that the common part of *A* and *B* is contained in *A*. Also $A \cap B \subset B$ and $A \cap B \subset A \cup B$. So the intersection is a subset of the union of *A* and *B*. Likewise I can write $A \cup B \supset A$ meaning that the union of *A* and *B* contains *A*.

The algebra of neural nets is couched in the notation of the propositional calculus so we will have to transfer our ideas into the isomorphic calculus. The symbols in the propositional calculus are called sentential connectives. Thus we have the isomorphism:

Algebra of Sets		Propositional Calculus
$\supset$ inclusion	⟷	$\supset$ implication
$\cup$ union	⟷	$\vee$ inclusive disjunction
$\cap$ intersection	⟷	( )( ) conjunction juxtaposition
= equals	⟷	$\equiv$ equivalence
CA in *S* complement	⟷	$\sim$ denial $\sim A$, not *A*

The biconditional $\equiv$ is interpreted like this: implication is called the conditional so $A \supset B$ or *A* implies *B* and $B \supset A$ or *B* implies *A* together mean that $A \equiv B$ or A is equivalent to B just as A contains B and B contains A means A equals B.

Householder and Landahl in their book Mathematical Biophysics of the Central Nervous System use the notation of propositional calculus in discussing neural nets after the manner of McCulloch and Pitts.

A neuron has an all or none character; either it will fire or it will not fire. Calling the afferent end the origin and the efferent end the terminus, it suffices to stimulate the origin beyond its natural threshold to cause the neuron to fire. During this active state a potential is created at the terminus which may in turn stimulate the next neuron in the chain. A fundamental property of the neuron is the fact that it must rest about one-half a millisecond before it can again fire. So we have a way of quantatizing time in terms of neurons. The natural interval of time is the interval between successive active states of a neuron.

Using a discrete measure of time we say an interval has been n, that is, n natural time intervals have elapsed. If we designate individual neurons by the symbols c_i we can let $N_i(t)$ denote the proposition that the neuron c_i has fired during the t^{th} time-interval. If this neuron is any but a peripheral afferent, directly fired by a receptor, then the necessary and sufficient condition for $N_i(t)$ is that a proposition or set of propositions of the form $N_j(t-1)$ shall be true; c_j being those afferents connecting with c_i.

In general there are two types of neurons, excitatory and inhibitory neurons. Also Rashevsky has proposed a mixed neuron combining both properties.

Taking our c_j to be excitatory we may suppose a set c_k which is inhibitory. Then $N_j(t-1)$ being true, $N_k(t-1)$ must be false. Let α_i be the class of subscripts representing any set of afferents which make excitatory connections with c_i and K_i the class of all such classes. Let β_i be the corresponding set for afferents making inhibitory connections with c_i. The negation of $N_k(t-1)$ is $\sim N_k(t-1)$ and the joint negation for all $k \epsilon \beta_i$ is written $\prod_{k \epsilon \beta_i} \sim N_k(t-1)$. The sufficient condition that c_i fire is $\prod_{k \epsilon \beta_i} \sim N_k(t-1) \prod_{j \epsilon \alpha_i} N_j(t-1)$. The necessary and sufficient condition is the disjunction of all such propositions

$\prod_{k \epsilon \beta_i} \sim N_k(t-1) \sum_{\alpha_i \epsilon K_i} \prod_{j \epsilon \alpha_i} N_j(t-1)$. Where Σ represents disjunction.

Defining a successor function or functor S by the equivalence $SN_i(t) \equiv N_i(t-1)$ we obtain a complete description of the activity of c_i.

$$N_i(t) \equiv S\prod_{k \epsilon \beta_i} \sim N_k(t) \sum_{\alpha_i \epsilon K_i} \prod_{j \epsilon \alpha_i} N_j(t).$$

Drawing a diagram where the triangles represent origins and dots termini or endfeet we indicate synapses by position.

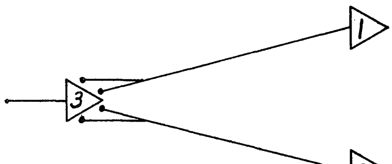

Figure 3

In this second figure c_3 having a threshold $\theta = 2\ c_3$ fires only if c_1 and c_2 have both fired during the preceding interval

$$N_3(t) \equiv SN_1(t)\ \ SN_2(t).$$

On the other hand with figure three where c_1 and c_2 each have two endfeet on c_3, c_3 fires if c_1 or c_2 has fired.

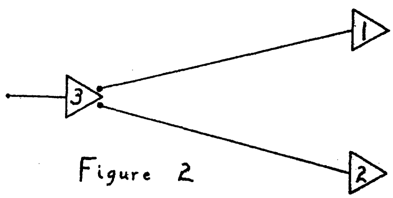

Figure 2

$$N_3(t) \equiv SN_1(t)\ \mathrm{v}\ SN_2(t)$$

If c_2 has only one endfoot and is inhibitory as in figure four:

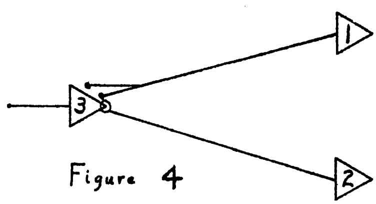

Figure 4

$$N_3(t) \equiv \sim SN_2(t)\ \ SN_1(t).$$

The right term of this equivalence is called a temporal propositional expression or *TPE*.

Figure 5 represents a neural net which McCulloch and Pitts have constructed to explain a phenomenon in which the illusion of heat is produced by transient cooling. A cold object held momentarily against the skin produces the sensation of heat whereas longer exposure gives only the sensation of cold. c_1 is the receptor for heat and c_2 is the receptor for cold while activity of c_3 gives the sensation of heat and of c_4 gives the sensation of cold.

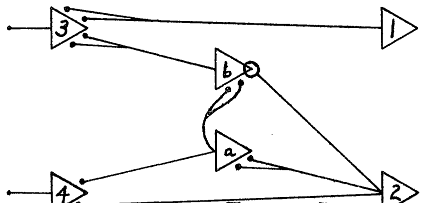

Figure 5

For the firing of c_3 either c_1 shall have fired or c_2 shall have fired momentarily only whereas for c_4 to fire it is necessary that c_2 shall fire for a period of time.

$$N_a(t) \equiv SN_2(t)$$

$$N_4(t) \equiv S[N_a(t) \quad N_2(t)] \equiv S[SN_2(t) \quad N_2(t)]$$

$$\textit{and } Nb(t) \equiv S[N_a(t) \sim N_2(t)] \equiv S[SN_2(t) \sim N_2(t)]$$

$$N_3(t) \equiv S[N_1(t) \vee N_b(t)] \equiv S\{N_1(t) \vee S[SN_2(t) \sim N_2(t)]\}$$

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A GEOMETRY OF CLOCKS

M. Y. Woodbridge

1. INTRODUCTION.

Keyser (4) called attention to the fact that the body of axioms and their subsequent theorems of geometry as set forth by Hilbert (1) may be considered to be one interpretation of a general form. To this form Keyser gave the name ''Hilbert Doctrinal Function''. To obtain other interpretations or doctrines of this same doctrinal function one has merely to replace the words ''point'', ''straight line'', ''plane'' and ''space'' by other sets of substantive terms with suitable definitions of the relational terms ''between'' and ''congruent''.

The present paper is an outline of an interpretation in which ''point'' is replaced by ''clock'', the term introduced by Kasner (3) to designate the graph of the first derivative of a polygenic (that is a non-monogenic) function for one value of the independent complex variable. This clock geometry is set up in a plane corresponding to Kasner's derivative plane.

2. DEFINITIONS AND DESCRIPTIONS OF ELEMENTS.

2.1 A clock is a circle with directed radius. We refer to a clock by naming its center. To avoid discussion of special cases we include in this definition clocks with radii of infinite length providing either center or end of radius is finite.

2.2 A unial is an infinite extent of clocks having a common line of centers, the ends of the directed radii being collinear and these radii prolonged being concurrent, (the common point to be called the radial point R). By allowing R to be either finite or infinite we include in this definition unials whose clocks have paralled radial directions.

2.21 The axis of a unial. Suppose we have a unial u with zero clock (that is clock of zero radius) at P (figure 3). Consider any pair of clocks A and B of u with ends of radii α and β respectively. Then since $A\alpha$ and $B\beta$ pass through the fixed radial point R, by a theorem proved by Johnson (2) the intersection of $A\beta$ and $B\alpha$ (call it S) must lie on a line through P for any choice of A and B. This locus of S is the axis of the unial.

Moreover, Johnson proved in the same theorem that the segment RS is cut harmonically by the two fixed lines (line of centers and line joining ends of radii). Hence the radius of each clock is cut harmonically by R and X (the point of intersection of the axis and radius). In case the

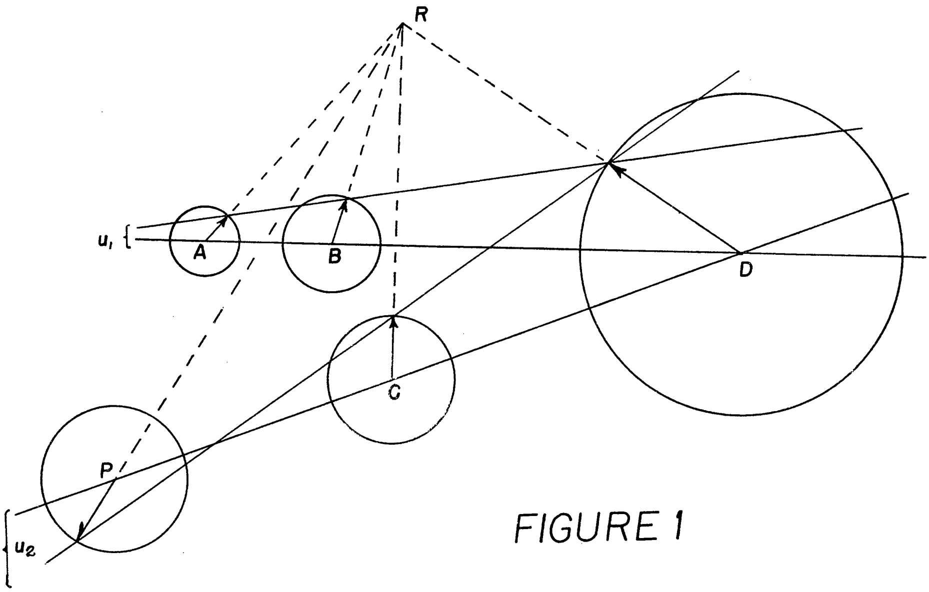

FIGURE 1

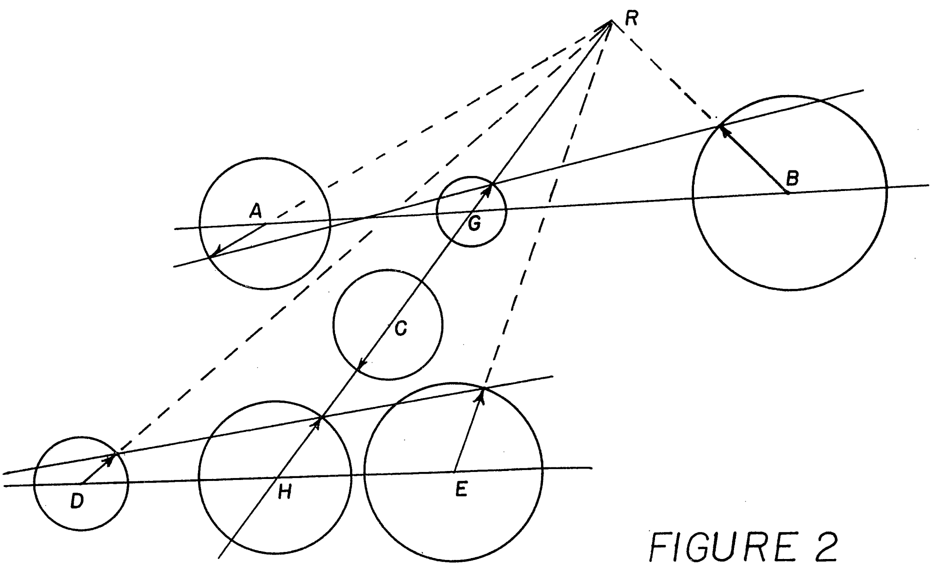

FIGURE 2

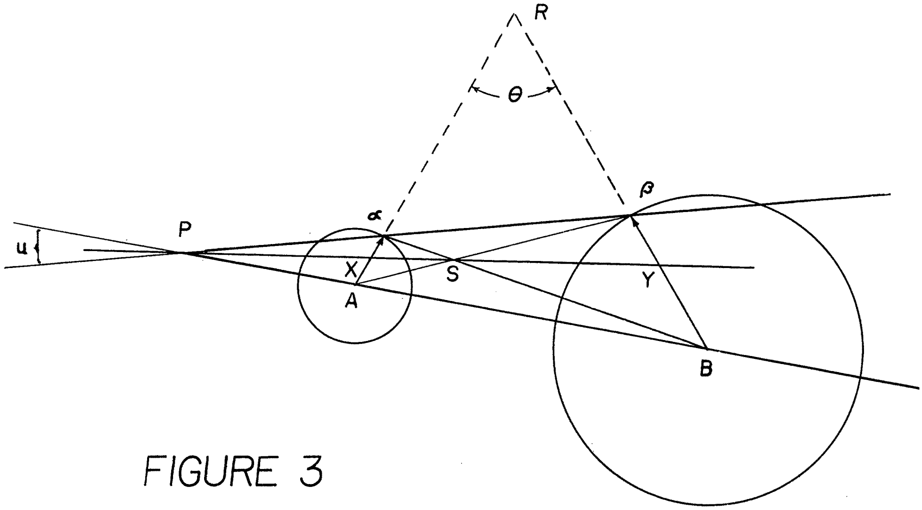

FIGURE 3

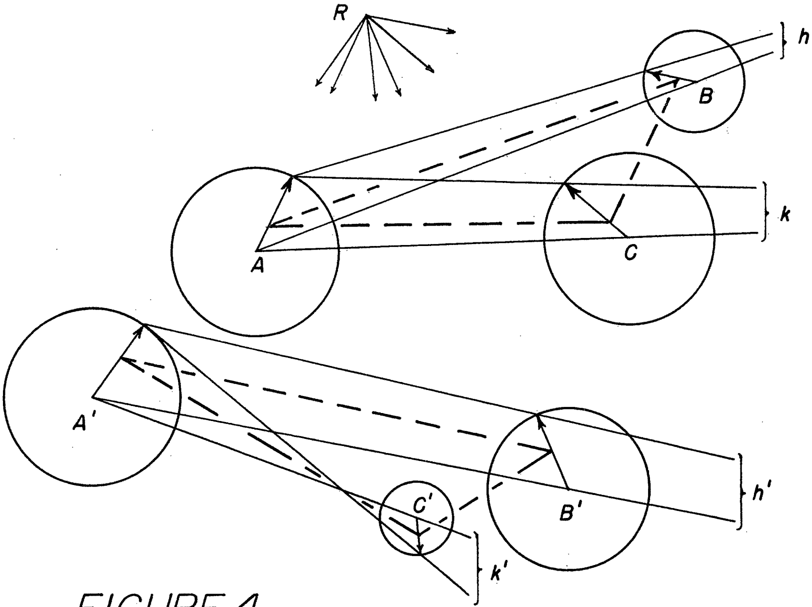

FIGURE 4

line of centers and the line joining ends of radii are parallel the axis is parallel to both lines. If the radii of the clocks of the unial are parallel the axis bisects the radii since R is now off at infinity and the harmonic ratio (αAXR) is $\alpha X/AX = -1$.

2.22 Note on some special forms of unials. We may have a unial of all zero clocks when the line of centers, line joining ends of radii and axis coincide. In this case though each radius be zero there is for each position on the common line a fixed radial direction. Again if one of the three lines pass through the radial point all three must coincide; the clocks of the unial then may or may not be zero clocks as will be seen in section 3.12.

2.3 A duplax is analogous to the plane of Euclidean geometry and admits of no concise definition. It may be visualized as an extent of clocks such that a unial two of whose clocks are contained in the duplax is wholly contained in it.

2.4 A radiam is the extent of all clocks having a common radial point.

2.5 The totum is the totality of all clocks in the plane. Since both length and direction of radius must be assigned after the center of a clock is fixed we see that the totum consists of ∞^4 clocks.

We limit our discussion to clocks, unials and duplaxes in a radiam and thus have a three dimensional geometry.

3. AXIOMS OF CONNECTION.

3.1 Two clocks of a radiam determine a unial. This is consistent with the definition of a unial, for given two clocks their directed radii pass through the radial point, the line of centers and line joining ends of radii are fixed.

Obviously if two zero clocks are given all other clocks of the unial determined by them must also be zero clocks. A similar statement can be made if the two given clocks have infinite radii.

3.11 One clock and the axis determine a unial. For lines through the end of radius and center of the given clock and any point on the axis will cut any line through the radial point repectively in the center and end of radius of another clock of the unial and the unial is then determined as described in 3.1.

3.12 Case where two centers and two ends of radii are collinear. We note that the anharmonic ratios of the centers of any four clocks of a unial are the same as the anharmonic ratios of the ends of the radii of these same four clocks since the two sets of points are perspective ranges with center of projection R. If we start with two clocks with centers and ends of radii collinear the unial must pass through the radial point R and furthermore the clock with center at R must have zero radius. Thus for any arbitrarily chosen center the anharmonic ratios of this center, R and the centers of the two given clocks are fixed and we know from projective geometry that only one end of radius can be found so that the anharmonic ratios of ends and centers shall be the same.

3.2 Similarly any two clocks of the unial determine it. The only exception to this statement occurs when one of the given clocks is the zero clock at R.

3.3 Three clocks A, B and C not in the same unial determine a duplax which is a two dimensional extent of clocks. By means of these three clocks we can fix a fourth clock at any point P by the following construction (figure 1). Join the radial point to P thus determining the radial direction; set up the unial u_1 through A and B; join P to the center of C and this line of centers will intersect u_1 picking out clock D at the intersection. Then C and D determine a second unial u_2 and the required clock at P must lie on u_2 thus the length of its radius is fixed. It may happen that radial direction of the clock at P is parallel to the line joining the ends of radii of u_1 then the clock P will have an infinite radius but such a clock is admissable from our definition in 2.1. Since D and therefore P both lie on a unial two of whose clocks are contained in the same duplax as A, B and C the clock constructed at P lies in the duplax by definition 2.3. Since the construction is unique there must be at each of the ∞^2 points of the plane only one clock belonging to the duplax and thus a duplax is a two dimensional extent of clocks.

This construction also shows that through a given clock in a given duplax only one unial can be drawn which has a prescribed line of centers.

3.4 Any three non-counial clocks in a duplax determine the duplax. A ready verification of this may be made by using three different clocks in the same duplax as A, B and C (to be sure they are in the same duplax they may be chosen from the unials joining pairs of these clocks) and demonstrating that the same clock is located at P by the construction described in 3.3.

3.5 If two clocks of a unial are in a duplax every clock of the unial is in the duplax. This was our description of a duplax in 2.3 and it was used to verify both 3.3 and 3.4.

3.6 If two duplaxes have one common clock they have another. To show this take A, B and C determining one duplax and C, D and E determining another and we may locate a second clock common to both duplaxes by the following construction (figure 2). First we notice that if the directed radius of C is excended through the radial point we have a unial which intersects the unial determined by A and B in clock G and G is known to belong to the duplax of A, B and C by section 3.5. Similarly the unial determined by G and C cuts the unial determined by D and E in a clock H which must then belong to both duplaxes by definition 2.3.

Thus two duplaxes which have a common clock intersect in a unial passing through the radial point.

3.7 From the above definitions and constructions it is apparent that we may say: every unial contains at least two clocks, every duplax at least three non-counial clocks and every radiam at least four non-duplaxal clocks.

4. AXIOMS OF ORDER.

4.1 If A, B and C are clocks of a unial and B is between A and C then B is also between C and A. This is a matter of definition of the word "between". We interpret this to mean that the center of B is between those of A and C on the line of centers and thus the axiom has the ordinary sense of point geometry.

In the same way the other three axioms of order hold.

4.2 If A and C are two clocks of a unial there is a clock B between A and C and a clock D such that C is between A and D.

4.3 Of any three counial clocks one and but one is between the other two.

4.4 Any four counial clocks can be so arranged that B is between A and C and between A and D and so that C is between A and D and between B and D.

4.5 Definition of a segment of a unial. A segment is determined by the segment of the line of centers and all clocks whose centers lie between the ends of the segment of the line of centers are called the segment's clocks.

4.6 Let A, B and C be non-counial clocks and let u be a unial of their duplax but not containing any of them. If u contains a clock of segment AB it also contains a clock of either segment BC or of segment AC. This is evident since the line of centers of u will cut either the segment between the centers of B and C or of A and C.

5. AXIOMS OF PARALLELS.

5.1 Definition of parallel unials. Two unials are said to be parallel if their axes are parallel.

5.2 If a unial u and a clock C not in u be in a duplax τ there is in τ

one and only one unial containing C but no clock of u. This unial through C is the one with axis parallel to the axis of u and since by 3.11 a unial of a duplax is fixed if one clock and the axis is known this unial through C is the only one parallel to u.

6. AXIOMS OF CONGRUENCE.

6.1 Definition of superposition. To superpose a clock A upon another clock B means to transform A continuously through the clocks which make up the segment AB. To superpose a segment $A'B'$ upon unial u, A' and B' are respectively superposed upon clocks A and B of u where the axis of segment $A'B'$ is equal in length to the axis of segment AB.

6.2 Definition of congruent segments. Two segments are called congruent if one can be superposed upon the other.

6.3 If A and B are two clocks of unial u and if A' be on unial u' then on either side of A' there is one and only one clock B' of u' such that segment AB is congruent to segment $A'B'$. (Every segment is congruent to itself).

6.4 If segment AB is congruent to segment $A'B'$ and to $A''B''$ then $A'B'$ is congruent to $A''B''$.

6.5 If segments AB and BC of a unial u have no common clock except B and if $A'B'$ and $B'C'$ of unial u' have no common clock except B', then if AB and BC are respectively congruent to $A'B'$ and $B'C'$, AC is congruent to $A'C'$.

6.6 Definition of half unial and angle. If A is a clock of unial u, the clocks of u on the same side of A constitute a half unial emanating from A. A pair of half unials h and k emanating from A form the angle (h,k); A is the angle's vertex and h and k the sides (figure 4); its interior is the class of clocks such that if M and N be any two of the class the segment MN contains no clock of h or k; its exterior is composed of all other clocks of the duplax except A and the clocks of h and k.

6.7 Definition of equal angles. Two angles with vertices A and A' are equal if the angle between the axes of the half unials meeting at A is equal to the angle between the axes of the half unials meeting at A'. We saw in 3.3 that through a given clock in a given duplax only one unial can be drawn which has a prescribed line as line of centers. And by definition of superposition in 6.1 one unial can always be superposed upon another. Hence two equal angles can always be make to coincide.

6.8 Given, in duplax τ, and angle (h,k), a clock A' and a half unial h' emanating from A'; then in τ and emanating from A' there is one and only one half unial k' such that angle (h',k') is equal to angle (h,k) and that the interior of (h',k') is on a given side of h' (figure 4).

To see that this statement is true we need only recall that from 3.11 there is only one unial through A' with a given axis.

6.9 If in triangles ABC and $A'B'C'$ (figure 4) AB, AC and angle BAC are respectively congruent to $A'B'$, $A'C'$ and angle $B'A'C'$ then the angles ABC and ACB are respectively equal to angles $A'B'C'$ and $A'C'B'$. That is the triangles of the axes are congruent in the ordinary sense of point geometry.

7. AXIOM OF CONTINUITY.

Let clock A_1 be chosen between any two clocks A and B of a unial u. Let clocks A_2, A_3, A_4, ... of u be such that A_1 is between A and A_2, A_2 between A_1 and A_3 and so on and such that segments AA_1, A_1A_2, A_2A_3, ... are mutually congruent. Then in the clock sequence there is a clock A_n such that B is between A and A_n.

8. DISTANCE.

From our definition of congruent segments in 6.2 it is logical to define the distance from clock A to clock B as the segment XY of the axis cut off by the radii of the two clocks (figure 3). We can obtain an expression for XY in terms of the two clocks, the radial point R and the angle θ subtended at R by the segment AB. First apply the law of cosines to the triangle XYR and get

$$XY = \sqrt{\overline{RX}^2 + \overline{RY}^2 - 2\,\overline{RX}\,\overline{RY}\cos\theta}$$

Then from the fact noted in 2.21 that $R\beta XB$ and $R\alpha XA$ are harmonic ranges

$$\frac{BY}{\beta Y} = \frac{-(\overline{RB}-\overline{RY})}{RY-R\beta} = \frac{-\overline{BR}}{\beta R} \quad \text{or} \quad RY = \frac{2\,\overline{R\beta}\cdot\overline{RB}}{R\beta + RB}$$

and similarly

$$RX = \frac{2\,R\alpha\cdot RA}{R\alpha + RA}.$$

Thus substituting these expressions for RX and RY in the formula for XY our definition for distance between clocks A and B is

$$2\sqrt{\frac{\overline{R\alpha}^2\cdot\overline{RA}^2}{(R\alpha+RA)^2} + \frac{\overline{R\beta}^2\,\overline{RB}^2}{(R\beta+RB)^2} - \frac{2\,\overline{R\alpha}\cdot\overline{RA}\cdot\overline{R\beta}\cdot\overline{RB}\cos\theta}{(R\alpha+RA)(R\beta+RB)}}$$

If the radial directions of A and B are parallel the form of this radical must be changed since R is now at infinity. Upon substituting 1 for $\cos\theta$ and taking the limit as $R\to\infty$ the result is found to be $\dfrac{(RA-RB)+(R\alpha-R\beta)}{2}$. This is half the sum of the projections of AB

and $\alpha\beta$ on a line in the radial direction which is exactly the projection of the distance XY in that direction in this special case where X and Y are the midpoints of the radii of the clocks A and B respectively.

9. CONCLUSION.

Since the elements clock, unial and duplax satisfy Hilbert's axioms we see that the geometry which follows is logically identical with Euclidean geometry. Thus our geometry of clocks in a radiam is a two dimensional picture of three dimensional Euclidean geometry. Similarly if all the clocks of the totum were considered we should have a plane representation of a four dimensional geometry.

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THE BASIC CONCEPTS OF TRIGONOMETRY

Estelle Mazziotta

Trigonometry might well be called the science of getting from *here* to *there,* especially when *there* is inaccessible. The problem of going from *here* to *there* resolves itself into two fundamental components: How far is it, and in what direction? From this it can be seen that trigonometry deals with lines (which measure distance) and angles (which indicate direction). The lines and angles are formed into the simplest of polygons, the triangle, hence the name "trigonometry", which really means "triangle measurement".

This branch of mathematics is the tool of the surveyor who measures land and computes the heights of mountains without scaling them, and also of the navigator (both sea and air) who sets courses and determines speeds. Trigonometry is one of the oldest branches of science and is associated with the surveying of land in ancient Egypt as well as with the development of both astrology and astronomy. Early surveying instruments, the forerunners of the modern transit, testify to this association by their elaborate decoration with the signs of the zodiac and representations of various constellations.

One of the basic problems of trigonometry is that of determining the height of an inaccessible object. Suppose, for example, that we wish to determine the height of a tree without being obliged to climb it. We can measure a convenient distance, say 40 feet, from the foot of the tree and, at this point, sight the top of the tree so that the line of sight completes a triangle that looks like Fig. 1.

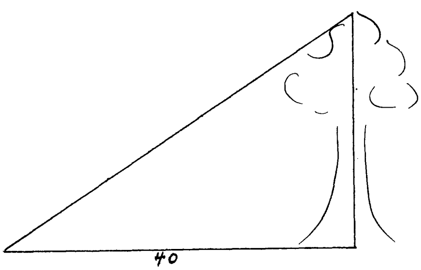

Figure 1

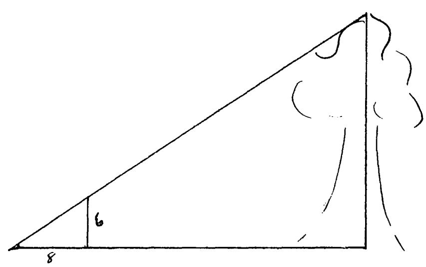

Figure 2

If a six-foot pole (see Fig. 2) is placed in such a way that its tip lies along the line of sight when its foot is perpendicular to the ground, we can compare the small triangle with the large and use this comparison to find the height of the tree, as we shall now show. Suppose the foot of the pole touches the ground at a distance of eight feet from the observer. It is apparent that the base of the small triangle is contained in the base of the larger one five times. Hence, if we fit a series of such small triangles along the line of sight as in Fig. 3, there would evidently be five of them. The base of the smaller triangle bears the same relationship to the total distance from the foot of the tree as the height of the pole bears to the height of the tree, that is, 8/40 is equal to 6/30.

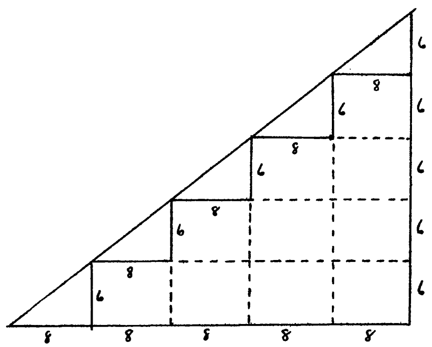

Figure 3

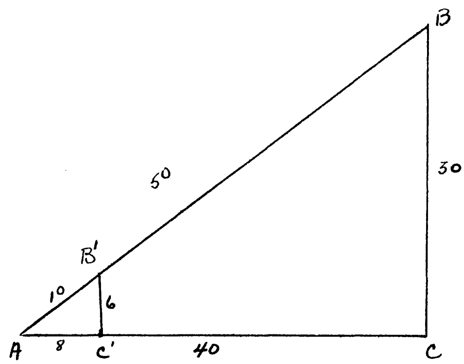

Figure 4

Such triangles are said to be similar, that is, alike in shape. *Their corresponding sides are proportional and their corresponding angles are equal.*

Our example serves as a key for understanding certain fundamental relations between the sides and angles of any right triangles which have a common acute angle. Consider the two triangles *ABC* and *AB'C'* in Fig. 4. Each side of triangle *ABC* is exactly five times as large as the corresponding side of triangle *AB'C'*. Another way to show this relation is to equate corresponding ratios in the two triangles as follows:

$$\frac{AC'}{AB'} = \frac{AC}{AB} \qquad \frac{C'B'}{AB'} = \frac{CB}{AB} \qquad \frac{C'B'}{AC'} = \frac{CB}{AC}$$

or, in terms of the above example:

$$\frac{8}{10} = \frac{40}{50} \qquad \frac{6}{10} = \frac{30}{50} \qquad \frac{6}{8} = \frac{30}{40}$$

These ratios are not dependent upon the area of the right triangle; they are dependent only upon the angle *A*. That is, they are the same for all such right triangles, $AB'C'$ and ABC, which have the common angle A. We call these ratios *Trigonometric functions* of angle A—trigonometric because they involve the sides of a triangle, functions of angle A because their values depend on the value of angle A. We name them as follows (see Fig. 5):

$$\frac{\text{side opposite}}{\text{hypotenuse}} = \frac{CB}{AB} = \text{sine of } A\text{, written } \sin A$$

$$\frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{AC}{AB} = \text{cosine of } A\text{, written } \cos A$$

$$\frac{\text{side opposite}}{\text{side adjacent}} = \frac{CB}{AC} = \text{tangent of } A\text{, written } \tan A.$$

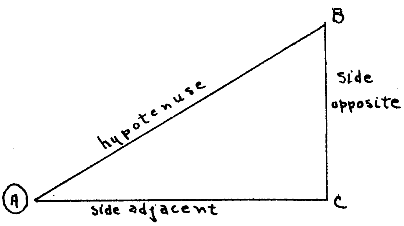

Figure 5

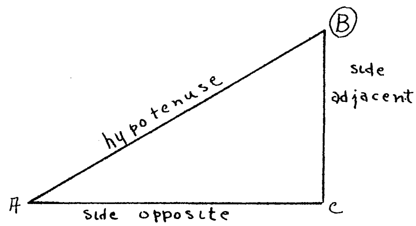

Figure 6

If we wish to list the trigonometric functions of angle B, we rename the sides of the right triangle with reference to B, as in Fig. 6. Then we find that $\sin B = \frac{AC}{AB}$, $\cos B = \frac{CB}{AB}$, and $\tan B = \frac{AC}{CB}$. Notice that the sine of A and the cosine of B are the same; likewise, the cosine of A and the sine of B are the same. Since A and B are the acute angles of a right triangle, their sum is a right angle, and they are said to be complementary. The word ''cosine'' implies ''complement's sine'', *i.e.*, the cosine of B is the sine of the complement of B, and vice versa.

It is possible, of course, to write three other ratios between the three sides of a right triangle taken in pairs. However, since these would be merely reciprocals of the ones already named, we mention them only for the convenience of the reader who may meet with them elsewhere. These additional functions are called cotangent, secant and cosecant and are respectively the reciprocals of the tangent, cosine and sine.

The variations in the values of the trigonometric functions of an angle which occur with changes in the angle can be demonstrated quite simply by placing the right triangle in a circle with radius one unit, in such a way that the radius appears in the denominators of the trigonometric ratios. Then we need observe only the changes which occur in single lines as the angle increases or decreases.

In Fig. 7, $AR = AP = AP' = 1$. In triangle APQ, $\sin A = \frac{PQ}{AQ} = \frac{PQ}{1}$; $\cos A = \frac{AP}{AQ} = \frac{AP}{1}$.

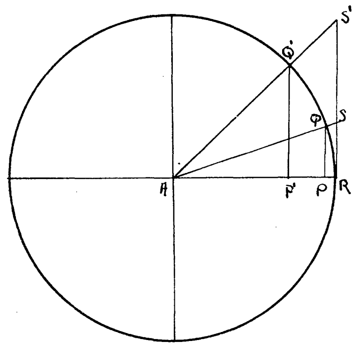

Figure 7

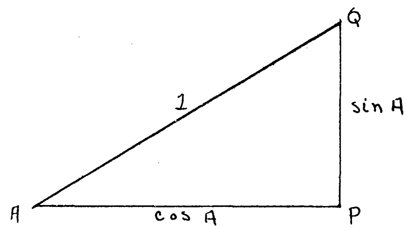

Figure 8

As angle A increases, PQ assumes a series of positions in the circle, one of which is shown by $P'Q'$. This indicates that the sine of an angle increases as the angle increases. It may also be seen from the diagram that $\sin A$ will reach a maximum value of 1 when A becomes 90°. Similarly, as angle A decreases PQ also decreases, and when A becomes 0° PQ reaches a minimum value of 0. In the case of the cosine, exactly the reverse is true. As angle A increases AP decreases, and when A reaches 90° the cosine, AP, becomes 0, and when A is 0° it will have a maximum value of 1. To observe corresponding changes in the tangent, it is necessary to consider another triangle, namely ARS. In this triangle, $\tan A = \frac{RS}{AR} = \frac{RS}{1}$. Like the sine, the tangent increases with the angle, but, as the figure shows, the increase is much more rapid. As the angle A approaches 90° this increase is without limit, and there is no tangent of 90° since this would involve division by 0.

The triangle in Fig. 8, which resembles the one in the unit circle, shows that the functions are so related to each other that given any one the other two can be determined. As an illustration we will find $\cos A$ and $\tan A$ in terms of $\sin A$. Since we defined $\tan A$ as $\frac{\text{side opposite}}{\text{side adjacent}}$, evidently, $\tan A = \frac{\sin A}{\cos A}$. It follows also, from the Pythagorean theorem, (which states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse) that

$$\sin^2 A + \cos^2 A = 1.$$

From this equation

$$\cos^2 A = 1 - \sin^2 A$$

whence

$$\cos A = \sqrt{1 - \sin^2 A}$$

Substituting the latter in $\tan A = \dfrac{\sin A}{\cos A}$ gives

$$\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$$

The *numerical values* of the functions of any angle could be approximated by means of a carefully constructed diagram, for instance, by measuring the lines *PQ, AP* and *SR* in Fig. 7. However, the values of the trigonometric functions are in general better approximated by means of series derived in other branches of mathematics. But the values of the functions of certain useful angles can be determined exactly from the properties of certain specific triangles. Let us consider, for example, an *isosceles right triangle,* that is, one having two equal legs. Suppose we denote the equal legs (Fig. 9) by any letter, say b; then we find the hypotenuse by the Pythagorean theorem: $c^2 = b^2 + b^2$, whence $c^2 = 2b^2$ and $c = b\sqrt{2}$.

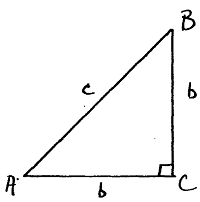

Figure 9

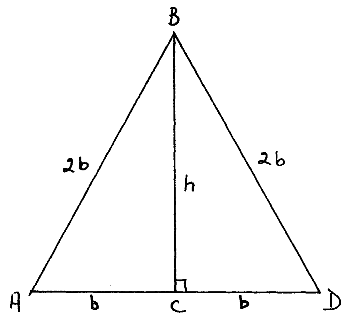

Figure 10

Since the sum of the angles of a triangle is 180^0 and the angles opposite the equal sides of a triangle are equal, each of the acute angles must be 45^0, so we can write the values of the functions of an angle of 45^0.

$$\sin 45^0 = \frac{b}{b\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.4142^+}{2} = 0.7071^+$$

$$\cos 45^0 = \frac{b}{b\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.7071^+$$

$$\tan 45^0 = \frac{b}{b} = 1.0000$$

In like manner, we can make use of an equilateral triangle--one having all sides equal and each angle 60^0 – to find the values of the functions of 30° and 60° (see Fig. 10). To avoid fractions, let us take each side of the triangle equal to $2b$ units in length. The perpendicular from B to the base bisects the base and the angle at B, producing right triangles having acute angles equal to 60° and 30°. The altitude, h, is found as follows:

$$h^2 + b^2 = (2b)^2$$

Transposing and combining like terms,

$$h^2 = 4b^2 - b^2 = 3b^2$$

Taking the square root of both sides,

$$h = b\sqrt{3}$$

Hence the functions have the following values:

$$\sin 30^\circ = \frac{b}{2b} = \frac{1}{2} = 0.5000$$

$$\cos 30^\circ = \frac{b\sqrt{3}}{2b} = \frac{1\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \quad \frac{1.7321^-}{2} = 0.8660^+$$

$$\tan 30^\circ = \frac{b}{b\sqrt{3}} = \frac{1\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.7321^-}{3} = 0.5774^-$$

$$\sin 60^\circ = \cos 30^\circ = 0.8660^+$$

$$\cos 60^\circ = \sin 30^\circ = 0.5000$$

$$\tan 60^\circ = \frac{b\sqrt{3}}{b} = \frac{\sqrt{3}}{1} = 1.7321^-$$

Now that we have learned something of the nature of the trigonometric functions, let us return to a consideration of our original problem involving the height of a tree.

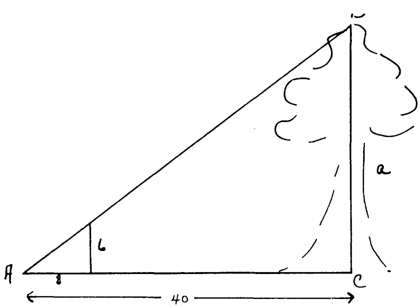

Figure 11

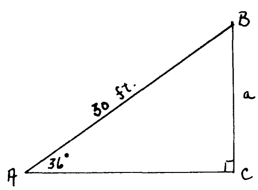

Figure 12

Using a transit (essentially an arrangement of telescope, and horizontal and vertical protractors), the surveyor would measure the angle at A (Fig. 11) by sighting the foot and top of the tree. He would find the angle to be about 36° 52'. Then he has the relationship

$$\frac{a}{40} = \tan 36^\circ\ 52' \cdot$$

From the tables he finds tan 36° 52' = .75
Whence,

$$\frac{a}{40} = .75$$

and

$$a = 40 \times .75 = 30$$

A variety of problems involving right triangles can be solved similarly. We cite a few interesting examples. (Note—Four-place tables were used in the solutions of these examples.)

I. How far up a wall will a 30-foot ladder (Fig. 12) reach if the foot makes an angle of 36° with the ground?

$$\frac{a}{30} = \sin 36^\circ$$

$$a = 30 \times 0.5878 \text{ or } 17.63 \text{ ft.}$$

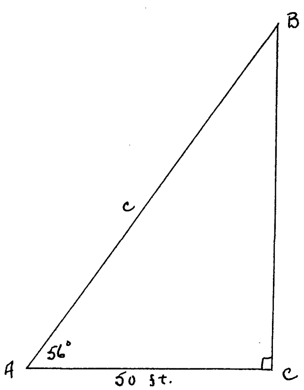

Figure 13

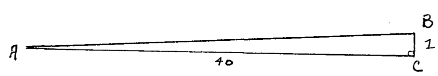

Figure 14

II. How long is the guy rope which supports a pole (Fig. 13), if the rope makes an angle of 56° with the ground and is fastened 50 ft. from the foot of the pole?

$$\frac{50}{c} = \cos 56^\circ$$

$$50 = c \times \cos 56^\circ = 0.5592\, c$$

$$\frac{50}{0.5592} = c$$

whence

$$c = 89.4 \text{ ft.}$$

III. What is the inclination of a plane (Fig. 14) which rises 1 foot in a horizontal distance of 40 feet?

$$\tan A = 1/40 \text{ or } .0250$$

$$A = 1^\circ\ 26'$$

Unfortunately, the conditions of a problem do not always lend themselves to a solution by means of a right triangle. Finding the height of a mountain, for example, may lead to a situation like that in Fig. 15 in which side c, and the angles of elevation at A and B can be measured: Obviously, some new method will have to be derived to find h.

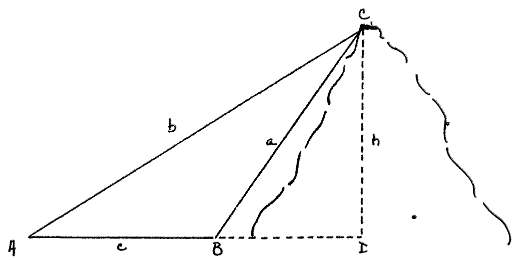

Figure 15

A nutcracker demonstrates the fact that, as an angle of a triangle changes, the side opposite that angle changes with it. Mathematically, we are concerned with the exact nature of this variation and it will be shown that the side increases or decreases in direct proportion with the sine of the opposite angle. From Fig. 15 we have, by the definition of the sine of an angle,

$$\frac{h}{b} = \sin A \quad \text{and} \quad \frac{h}{a} = \sin B$$

Whence

$$h = b\sin A \quad \text{and} \quad h = a\sin B$$

Equating these two values of h, we have

$$b\sin A = a\sin B$$

Dividing both sides by $b\sin B$ gives

$$\frac{b\sin A}{b\sin B} = \frac{a\sin B}{b\sin B}$$

Cancelling out like terms in numerators and denominators, we obtain

$$\frac{\sin A}{\sin B} = \frac{a}{b}$$

Stated in words: *The sides of a triangle are proportional to the sines of the opposite angles.* In its complete form the Law of Sines may be written:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now let us proceed with the solution of the problem of Fig. 15. Suppose, upon measurement, the angles of elevation at A and B prove to be 34° and 50°, respectively, and side c measures 3000 ft. Then angle ABC is $180^\circ - 50^\circ$ or 130° and angle $C = 180^\circ - (130^\circ + 34^\circ) = 16^\circ$. Applying the Law of Sines, $\frac{a}{\sin A} = \frac{c}{\sin C}$, we have in this problem

$$\frac{a}{\sin 34^\circ} = \frac{3000}{\sin 16^\circ}$$

Multiplying both sides by $\sin 34^\circ$, and using a four-place table, we obtain

$$a = \frac{3000 \times \sin 34^\circ}{\sin 16^\circ} = \frac{3000 \times 0.5592}{0.2756} = 6087$$

In the right triangle BCD, angle B is 50° and $a = 6087$. Whence

$$\frac{h}{6087} = \sin 50^\circ$$

$$h = 6087 \times \sin 50^\circ = 6087 \times 0.7660 = 4663 \text{ ft.}$$

Were we required to find side b in the oblique triangle just considered, application of the Law of Sines would necessitate finding the value of the sine of angle ABC, that is, $\sin 130^\circ$. This suggests that we need to generalize the idea of trigonometric functions of angles to include angles greater than 90°. In fact we consider trigonometric functions of angles even greater than 180°. Although angles greater than 180° cannot be used in triangles, they are met in problems involving rotary motion.

Since the unit circle shows continuous changes in the functions as the angle increases toward 90°, we again turn to this device to build our concept of functions of angles greater than 90°. For convenience we adopt the mathematical convention of dividing the plane into four quadrants by means of a pair of perpendicular lines called axes, (see Fig. 16) and of regarding distances measured upward, and to the right, as positive, and those measured downward, and to the left, as negative. As before (in Fig. 7), PQ represents $\sin A$ and we generalize our definition of the sine to be the perpendicular distance to the horizontal axis from the point at which the terminal side of the angle touches the unit circle.

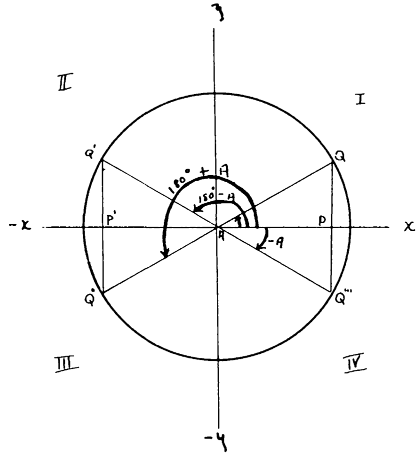

Figure 16

Then the lines corresponding to PQ in the other quadrants represent the sines of angles as follows:

$P'Q' = \sin PAQ' = \sin (180^\circ - A)$

$P'Q'' = \sin PAQ'' = \sin (180^\circ + A)$

$PQ''' = \sin PAQ''' = \sin (360^\circ - A)$

But lines PQ, $P'Q'$, $P'Q''$, and $PQ'''$are all equal in length. Some of them, however, differ in direction, thus

$P'Q' = PQ$

$P'Q'' = -PQ$

$P'Q''' = -PQ$

Therefore, it is possible to express the sines of angles in the second, third and fourth quadrants in terms of the sine of a related angle in the first quadrant. These relations follow:

Quadrant II $\quad \sin (180^\circ - A) = P'Q' = PQ = \sin A$

Quadrant III $\quad \sin (180^\circ + A) = P'Q'' = -PQ = -\sin A$

Quadrant IV $\quad \sin (360^\circ - A) = PQ''' = -PQ = -\sin A$

We list the following numerical examples (using a four-place table):

$$\sin 130^\circ = \sin (180^\circ - 50^\circ) = \sin 50^\circ = 0.7660$$

$$\sin 230^\circ = \sin (180^\circ + 50^\circ) = - \sin 50^\circ = - 0.7660$$

$$\sin 310^\circ = \sin (360^\circ - 50^\circ) = - \sin 50^\circ = - 0.7660$$

By means of the same device, similar relations can be derived for the cosine and tangent.

We now have at our command sufficient knowledge of trigonometry to appreciate and solve problems in some of the most important fields of applied mathematics. In navigation, one meets such problems as the following:

> A pilot wishes to make good a course in the direction 220° (with the northern direction). A 25 m.p.h. wind is blowing from 80°. If his air speed is 200 m.p.h., in what direction must he head the plane and what will be the ground speed?

By use of the sine law and a table of the values of the trigonometric functions one calculates the desired direction to be 215° 23′ and the desired speed to be 218.5 m.p.h., but we shall omit the details here.

A further use of trigonometry becomes apparent from a geometrical representation (graph) of the equation $y = \sin x$ on a pair of perpendicular lines as indicated in Fig. 17. When a value is assigned to x, the corresponding value of y can be determined from triangles or from tables. For example, if $x = 30^\circ$, then $y = \sin 30^\circ = .5$. Other pairs of values are shown below:

x	0°	30°	60°	90°	120°	150°	180°	240°	270°	300°	330°	360°
y	0	.5	.866	1	.866	.5	0	−.866	−1	−.866	−.5	0

Plotting these pairs of values and joining them with a smooth curve, the graph of $y = \sin x$ takes the following form:

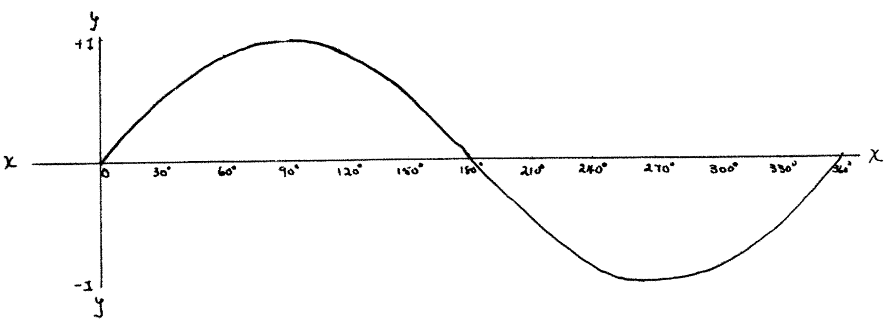

Figure 17

If we were to graph $y = \sin 2x$, the loops would be half as long, *i.e.*, there would be twice as many of them. Again, the graph of $y = 2 \sin x$ would differ from the graph of $y = \sin x$ in that it would vary from a maximum height of + 2 to a minimum of − 2, that is, the highest and lowest points of the curve would be twice as far apart as those on the graph of $y = \sin x$. In general, in the graph of $y = a \sin bx$, a is called the *amplitude* of the curve, and b the *frequency*. This terminology suggests that these curves are associated with the study of wave phenomena—light, sound and electricity. In fact, the curves traced by the needle on a speech-recording machine are sine curves.

While we have not, by any means, covered the solution of all types of triangles nor all the theory of trigonometry, we believe that this brief survey is sufficient to indicate that trigonometry has come a long way from the pseudoscience of astrology and that it occupies an honorable place among the branches of mathematics.

University High School
Los Angeles, California

INQUIRIES IN MATHEMATICS

Bancroft H. Brown

The old New England colleges knew a great deal about general education. That was, in fact, the only kind of education they knew anything about. Every student took exactly the same courses as every other student. Usually those courses had remained substantially the same for generations. At Dartmouth College, from 1769 to 1881, over a period of 112 years, the only real change in the mathematics was that in 1833 the Trustees finally secured a teacher who, in theory, could teach the calculus.

Gradually, and despite the violent protests of the teachers of the classics and mathematics (the ones who had gotten in on the ground floor) other subjects were added, and eventually there were so many subjects that some election was necessary. The principle of election was conceded grudgingly by most. It was, for a few, the guiding star to a shining new world. The classics lost their fight for survival, mathematics went down, all disciplines and cultures as requirements seemed to be lost - but as the last few grains of the required sands fell into the lower half of the hour-glass, we find a few brave souls who would invert the glass and even plug the central hole so that everything in the future could be saved.

The College of the University of Chicago, for example, now has leaders as omniscient as were President Dunster of Harvard, Eleazar Wheelock of Dartmouth, and Mary Lyon of Mount Holyoke. They know what is best for the student; they know that what is best for one is best for all; and they do not hesitate to tell us that we are derelict to our duty if we donot accept their principles, and climb on their band-waggon.

So the cycle has come full turn, and many of our educational leaders accept the principles of 100 years ago. But with this difference.

One hundred years ago, Harvard and Yale, Princeton, and Dartmouth, and Mount Holyoke, all taught the same subjects, and taught them in about the same way. They never will again. They will differ as to what an educated man or woman should know; and their differences will not be reconcilable, and their differences will probably increase.

In the Report of the Harvard Committee, we find that "General Education" is used to indicate that part of a student's whole education which looks first of all to his life as a responsible human being and citizen; while the term, special education, indicates that part which looks to the student's competence in some occupation.

But notice there is no immediate claim by the Harvard Committee that the education of the student to make him a responsible human being and citizen is an absolutely prescribed education. The Committee is simply saying: "We didn't do enough of that kind of thing in the past, we must do it in the future." But there is no required program. There is simply recognition that there is a job to be done.

Chicago, on the other hand says, "We agree, there is a job to be done. And until it is done, special education must wait in abeyance". And further, Chicago would say that there is a best way of doing it, and what is best for one, is best for all.

I think it is important for me to say that I am apposed to this principle. I think it much less important that I am opposed to the particular course which Chicago has set up for "best". The Chicago course in mathematics is 1/5 logic; 2/5 algebra; 2/5 trigonometry and analytic geometry. I doubt the value of much of the algebra; I doubt the value of a fair part of the trigonometry and analytic geometry; and I would venture the categorical statement that a year-course in mathematics in a college of liberal arts which did not include the fundamental concepts of the differential and integral calculus was per se a wasted effort.

The Chicago course isn't the only general education. Dozens of "survey courses" have been devised, containing a melange of algebra, trigonometry, analytic geometry, and even the calculus; together with logic, history, and philosophy. The proponents of these survey courses claim that their course will do two things:

(1) it will be better for the non-scientist who goes no farther;

(2) it will be better for the future specialist in science.

I doubt if you can have it both ways; and for my money these people lose out both ways.

And I am not in the least impressed when they say (as they always do) that *this* course is motivated and integrated and unified by the *function concept*.

Now you are going to hear about two different types of course in "general mathematics": (1) the Dartmouth course; (2) the Mount Holyoke course.

Dartmouth would say:

(1) we can give a valuable course in mathematics for men who are interested in science and who will probably major in science, and who need mathematics as a tool, as well as a humanizing discipline.

(2) we can give a different kind of course for men who are not particularly interested in science, but who can profit from the discovery of the impact of science on humanity.

Mount Holyoke would say:

We think we can do better than that. We think we have a course which from the point of view of general education actually does look to the development of a student as a responsible human being and citizen; and also does add materially to the student's competence if he chooses to work in science as his special field.

Before I describe the new course in general mathematics which I have introduced at Dartmouth, I should like to add these further comments.

First. The Mount Holyoke course is the most intelligent, realistic attempt to do an admittedly very difficult thing that I have ever seen. These people are good, they know what they want to do.

Second. There are possibly some slight elements of sex warfare in all this. I should hesitate to try the Mount Holyoke course at Dartmouth (altho some of my colleagues do not share my doubts). It might well be that the Dartmouth course would not be favorably received on a girls' campus.

Third. Secondary school teachers might as well reconcile themselves to the realistic fact that for years to come, college courses are going to vary in a most incredible way, and it is not going to be possible for them to prepare students for a particular kind of course.

Mathematics 1. Inquiries in Mathematics.

This course is especially designed for the student who wishes to know something of the significance of mathematics in the world of to-day but who does not wish to devote more than one semester to mathematics. The course does not include a systematic treatment of any brance of mathematics, nor is it a survey course. It selects several problems of importance to mankind; shows what foundation is required, what techniques need to be developed, and the nature and validity of the results that can be obtained, together with a view of the extensions of the problem. A major portion of the course will be devoted to problems whose solution requires the study of the fundamental methods of the calculus. The remaining time will be spent in the study of two or three topics such as probability, cartography, real number system, etc.

Open to students who have had Elementary Algebra and Plane Geometry. This course taken alone can be counted toward fulfillment of the science requirement for the bachelor's degree; but Mathematics 1 together with Mathematics 3 cannot be counted as a year of science. Students should not elect this course if they plan to take any further work in mathematics, or if they plan to major in any science.

With regard to this new course (for which I take some credit, and all the blame), the *first* thing to note is that we pass up completely the idea that it prepares in any way a man for more advanced courses. We tried, in fact, to formulate a transition course to enable a man to take Math. 1, then the transition course, and then go on with the sophomore calculus, and we failed utterly. So this course is for the man who intends to take this and nothing more. In the few cases where this course inspires a man to go on with mathematics, he will have to begin all over again in the traditional sequence.

Once we have accepted, for better or worse, this first fact, we now have the *second* fact that we are under no pressure to include any subject whatsoever, nor to give any systematic treatment of trigonometry, or of conic sections, or of crap-shooting, unless we want to. I can hardly overestimate the freedom from pressure groups that we have obtained by this decision, which was so simple, and yet so hard to make. For instance, the Commanding Officer of the NROTC says he would be glad to require this course for men under his jurisdiction if we will include a minimum of 4 weeks of trigonometry; and we don't want 4 weeks of trigonometry, and we say NO. Independent of the U.S. Navy, the U.S. Chamber of Commerce, (the Dies Committee), and the U.S.S.R., we are free to pick the subjects that we think make for the better general education of the citizens of this country. We think cartography and probability are such subjects. For different reasons, but with equal validity we would like to have our students know something of our number system: what it is, why it works, and what it can do. And this brings us to: —

The *third* fact, that we really do want a certain amount of systematic development in this course, and we want it devoted entirely to the calculus. The technique involved - and it amounts only to the differentiation and integration of polynomials, is really not extensive. It can be developed in a week. The problems here are of real importance: motion, and summation are at the root of a great deal of our modern quantitative thinking. The fact that about one-half of the course actually hangs together, I do not consider detrimental; and of course the disassociation of the rest of the course does not worry me a bit.

A *fourth* fact is that the course is not designed to be a disciplinary course. It bothers me a little to put it so bluntly, and yet that is the fact. Can I assume that we talk the same language when I use the word *disciplinary*? I use the word in the sense in which Latin, Greek, and mathematics were taught in the better academies 40 years ago. It involves the dictionary definition of descipline: "Training which corrects, molds, strengthens, or perfects". But discipline, as I knew it, was more than that. Discipline implied that only a very few would survive - and that they would, from their discipline, be a race apart. I recall with great distinctness a saying of a disciplinarian of the old school; —

"Young gentlemen, there are two reasons for studying Greek.

The first is so that you can read the New Testament in the original Greek.

The second is so that you may have a proper contempt for those who cannot."

Now I am a product of the old school, and I believe firmly that if any man - God help him - wants to be a mathematician, he must subject himself to a Calvinistic discipline which, if anything, increases as time goes on. But - and my metaphors are going to get badly mixed in a moment - 99% of the students who go to college are not going to be mathematicians, and it is my duty to be a faithful shepherd to the ninety and nine. As for the hundredth sheep, he has just got to find his own path for himself and I won't even lift my crook to help him. The only thing to do is to put a few more obstacles in his way and see if he has the ability to turn from a sheep into a goat and climb over them. So in this course which I have planned, homework is occasional, rather than regular. In my course discipline is not in the forefront.

And as a *fifth* fact, let me repeat what I said earlier. It is not contemplated that every student should take mathematics: this course or a more traditional sequence. I do not, and will not accept the full implication of "general education" as some people use it: that is, a body of learning which ALL college graduates must have in common. I recognize that colleges should have a science requirement (personally I think they put this requirement in general too high). I believe that such a course as I have described is a good course for a large number of college students; and that it should count in partial fulfillment of such requirements. And if you want a prediction, I would say briefly, that of 7 college students:

3 should take general education in mathematics
2 should take a traditional course in mathematics
2 should not take any mathematics.

I am overdoing the classification feature in this summary, but I should like to stress a final feature: the *sixth* I think it is by this time, and that is the informational character of this course. For this is not a course in pure mathematics. In teaching it, and in taking it, teacher and pupil have to have a common knowledge of a lot of things. As far as the pupil is concerned, he gets much of this knowledge presently from his teacher; and it is temporary, does not involve memorization, and will naturally slip after the course is over. The teacher - Heaven help him - has got to know this supplementary data cold.

In cartography there must be an appreciation of the historical growth of knowledge of geography, and that implies names and dates and facts. There must be recognition of the present importance of the north polar region, and that means more than details of map projections: it means knowledge of Greenland, Ellesmere Land, Spitzbergen, and Novaya Zemlya.

In probability I do not know how the inclusion of the subject can be justified unless we frankly (and realistically and intelligently) discuss such bizarre subjects as roulette, crap-shooting, pari-mutuel horse-racing, lotteries, and the numbers racket. These are subjects such that a law-abiding mathematician may well shrink from acquiring first-hand information as to their inner workings.

The Mt. Holyoke course and the Dartmouth course overlap on the real number system, altho they do a much more thoro job. Both they and we are under obligations to exhibit peculiar systems for which our postulates hold; and reasonable, altho exotic systems, for which they do not hold.

The calculus is no exception. If you are going to teach general mathematics in any college in these United States, you just have to know a large amount of miscellaneous information of a quantitative nature. This type of information is not of the type included in the graduate schools of Harvard, Princeton, Chicago, and California. It is hard come by.

I return to the original idea of my course. The human race has asked itself certain questions: questions such as "Where, When, How Fast, How much, What's the Chance?" Mathematical disciplines have been devised to answer these questions. I want my students to know the kind of thing we have been able to do in the past. I want them to share my faith that when new problems come up we shall be able to find some of the answers.

Dartmouth College.

Prof. Joseph Seidlin made the following interesting comments when accepting the above paper for his department. Ed.

As for my own reactions to the new Dartsmouth course, they are "mixed". I am all for proper kind of experimentation with the content of *any course*. That goes even for the so well entrenched pre-requisites. Nor am I concerned with the specific nature of the chosen topics, viz., probability, differentiation and integration of polynomials, our so pretty number system (Ginsburg of Yeshiva College tells us that mathematics can be beautiful). That reminds me of a recent experience in a nearby school. An extremely homely spinster, one of the older teachers, passed by while I was engaged in conversation with a nine-year-old. When the teacher was out of ear-shot, the child remarked, "Isn't she the most beautiful person you ever saw?" Pretty soon a younger, rather "pretty" passed us, and the same child said, "She is the droopiest teacher in the school." So with mathematics: very beautiful to some; droopy to others. But, they tell me, the right kind of clothes, hairdo, lipstick, and make-up can work wonders for women. I know that the right kind of teaching can make even trigonometry "fascinatin' "; the wrong kind of teaching can make even the "laws of chance" droopy.

I do believe, and very strongly, that whatever the content (within reasonable limits, naturally) it can be made meaningful and exciting to most normal students. As for preparing them for more advanced work, it isn't the quantity of content in any given course, but the quality of treatment (and learning) that makes a student "readier" for the next course.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

The purpose and policies of the first division of this department (comments on Papers) derive directly from the major objective of the MATHEMATICS MAGAZINE which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy will be sent to the author of the original article for rebuttal.

Communications intended for this department should be addressed to

H. V. Craig, Department of Applied Mathematics,
University of Texas, Austin 12, Texas

Les Grands Courants de la Pensée Mathématique. Textes recueillis et présentés par F. Le Lionnais.

This volume of 550 pages (840 francs) is essentially a collection of about fifty articles written by some forty-five leaders in various branches of mathematics and the allied domains. The first major division of the book bears the title *Le Temple Mathématique* and is divided into two parts: *A*—Structures, *B*—Disciplines. Some of the topics treated are: definitions, analogies, symmetry, the architecture of mathematics, the intuitive paths to the essential features, natural numbers and their generalizations, Fermat's last theorem, the history of Π, e, C, and i, transfinite numbers, abstract spaces, the fourth dimension, curvature of space, various aspects of the theory of functions, groups,, and probability.

Division II, *L'Epopée mathématique:* A—Passé, B—Présent, C—Avenir presents articles on: the evolution of mathematics, Newton, Lie, mathematics at the beginning of the century, Lebesque, Hilbert, the International Congress, and modern methods and the future of mathematics.

Division III, *Influences* contains nineteen papers classified under the headings: A—Role des Mathématiques dans la Formation de L'Esprit Humain, B—Les Mathématiques et la Philosophie, C—Les Mathématiques, la Vérité, la Réalité et les Sciences de la Nature, D-Les Mathématiqees, la Beauté, l'Eshhétique et les Beaux-Arts, E—Les Mathématiques, l'action, les Techniques et la Domination de la Nature, F—Role des Mathématiques dans la Société et dans le Développement des Civilisations. Some of the subjects treated are: ''La position moderne du debat: esprit géométrique, esprit de finesse'', Mathematics as an object of culture and as a tool, logical synthesis of researches, the duality of the philosophy of mathematics, mathematics and philosophical idealism, mathematics and Marxism, the position of mathematics in the classification of the sciences, the role of mathematics in the development of modern physics, classical and modern physics, the effect of physics on mathematics, beauty and mathematics, the group concept and the arts, architecture and mathematics, mathematics and music, mathematics in industry, mathematics and social development.

The authors are: Jean Ballard, Marcel Boll, Emile Borel, Louis de Broglie, Georges Bouligand, Nicolas Bourbaki, Leon Brunschvicg, Pierre Brunet, Adolph Buhl, Elie Cartan, Jacques Chapelon, Le Corbusier, Robert Deltheil, Arnaud Denjoy, Jean Desanti, Jean Dieudonné, Mme Marie-Louise Dubreil-Jacotin, Rene Dugas, Henri Eyraud, Robert Fortet, Maurice Frechet, Paul Germain, Lucien Godeaux, Théophile Got, Maurice Janet, Theo Kahan, Paul Laberenne, Albert Lautman, Andre Lantin, Francois Le Lionnais, Michel Luntz, Henri Martin, Paul Montel, Paul Mouy, Louis Perrin, Raymond Queneau, Jacques Reinhart, Maurice Roy, André Sainte-Lague, Pius Servien, Andreas Speiser, René Thiry, Jean Ullmo, George Valiron, Paul Valéery, Rolin Wavre, André Weil.

University of Texas — Homer V. Craig

Modern Operational Calculus by N. W. McLachlan xiv + 218 pages, Macmillan and Co., Ltd., London, 1948. 21 shillings.

This book provides a concise introduction to the theory and application of Laplace Transforms.

Content: The book opens with the definition of the p-multiplied Laplace Transform (LT), and then it states Lerch's uniqueness theorem. Conditions are given for the absolute and uniform convergence of the Laplace Integral, and an application is made of the property of uniform convergence to the evaluation of $\int_0^\infty f(t)dt$. The second chapter lists the LT's of various functions involving $f(t)$. The next three chapters give applications of LT's to: (a) the solution of ordinary linear differential equations, with illustrations from simple electrical networks; (b) the solution of partial linear differential equations, with illustrations from heat conduction, wave motion, and electric cables; and (c) the evaluation of definite integrals and the expansion of functions. The last two chapters develop the LT's of certain special functions, LT's for a finite interval, and a theory of impulses. The book closes with a 24 page collection of problems

Style: The principal emphasis of this book is on rigor. The author points out that ''In enunciating a theorem, the conditions for its validity must be stated fully. Otherwise the theorem might be used in cases where it did not hold. Sometimes a theorem may hold under less stringent conditions than those given in the proof. • • • Omission of logical steps in analysis is just as serious a defect as absence of credits in a cash account. The answer to a problem is essential, but *its correctness is imperative.*'' The author includes in each theorem a sufficient condition, and justifies each major step of proof by citing a theorem of analysis. Certainly the logic of the presentation is excellent.

However, the author's style is very mechanical. Motivation and heuristic are almost completely lacking, and the student is treated as little more than a machine for performing LT's. Chapter II, for example, may almost be summarized by the formula: theorem, proof, example; theorem, proof, example.

It is assumed that the reader is familiar with Bessel functions and with the concepts and methods of analysis (except in Chapter I where for some reason the very simple ideas of boundedness and continuity are described). But for the reader who is not disciplined in analysis, there are included a series of appendices listing the concepts and theorems needed—with numerous examples, but ''to economise in space, proofs of the theorems are omitted''. It is difficult to diagnose the author's pedagogical standpoint.

Some of the mathematical conventions in this text may be confusing to those who are not already familiar with their intended meanings. For example, at the bottom of page xiii the author states that ''throughout the text a continuous function means one which is differentiable.'' He goes on to say that ''There are, however, certain continuous functions (not contemplated herein) which are not differentiable'', but it is difficult to see what is gained by the switch of standard terms. Again, on page xii the author states that ''$f(t) = O(1)$ means that the function is bounded. When $t \to +\infty, \cdots, t^2/(1 + t^2) = O(1)$ with bound unity. When $t \to 0$, $(t^2 + a^2) = O(1)$; $\sin t = O(t)$.'' Is the first sentence of of this last quotation a definition or a theorem? And is the function ''bounded'' or ''bounded over a neighborhood of the limit value of t''?

Conclusion: This book is an excellent reference by an outstanding mathematician. But in its present form it would seem that the finesse of the presentation is unnecessarily inaccessible. In a text, the verbal acts should be designed with the same precision as the logic.

C. C. Torrance

Advanced Calculus, David V. Widder, New York, 1947, Prentice-Hall.

According to the author's preface, ''This book is designed for students who have had a course in elementary calculus covering the work of three or four semesters. However, it is arranged in such a way that it may also be used to advantage by students with somewhat less preparation.'' The chapter headings are as follows: I. Partial differentiation, II. Vectors, III. Differential Geometry, IV. Applications of partial differentiation, V. Stieltjes Integral, VI. Multiple Integrals, VII. Line and Surface Integrals, VIII. Limits and Indeterminate Forms, IX. Infinite Series, X. Convergence of Improper Integrals, XI. The Gamms Function. Evaluation of Definite Integrals, XII. Fourier Series, XIII. The Laplace Transform, XIV. Applications of the Laplace Transform.

Perhaps the most striking feature of the book is the concise way in which definitions and theorems are stated. For example (from Chapter VIII):

Theorem 1. 1. $f(x),\ g(x) \in C^1 \qquad a \leqq x \leqq b$

2. $f(c) = g(c) = 0, \qquad a < c < b$

3. $g'(c) \neq 0$

$$\rightarrow \quad \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} .$$

The proofs are likewise concise. The author frequently postpones the consideration of certain points in the proofs of early theorems until later on in the book. For instance, in the proof of Rolle's theorem (p. 7) he says, ''If $f(c) > 0\,(<0)$, then $f(x)$ has a maximum (minimum) at a point ξ, $a < \xi < b$.'' In a footnote he says, ''This fact is obvious geometrically. A proof by use of Definition I [continuity] alone will be found following Theorem 7 of Chapter V.''

Formalism is somewhat accented; the inner meaning of the theory is not emphasized. This is illustrated by the theorem quoted above and its proof. The author uses the law of the mean in the proof. Of course, under a much weaker set of hypotheses,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\dfrac{f(x) - f(c)}{x - c}}{\dfrac{g(x) - g(c)}{x - c}} = \frac{f'(c)}{g'(c)} .$$

There are numerous exercises, for instance, 107 in Chapter I, 68 in Chapter II, 79 in Chapter III. Many of these are formal in character. The book is rich in illustrative examples.

Concerning the treatment of integration, we quote from the author's introduction to Chapter V: ''The student is assumed to be familiar with the ordinary theory of the definite integral. The Stieltjes integral is, however, only a slight generalization of that familiar integral, so that what follows may be used by him as a review or solidification of the classical theory.'' There follows a definition of Stieltjes integral, proof of existence for a continuous function with respect to a monotone function, properties of the integral, integration by parts, laws of the mean, and physical applications. The last eight pages of this chapter contain fundamental properties of continuous functions, whose consideration was postponed until this point. Chapters VI and VII contain an adequate treatment of multiple integrals and of line and surface integrals.

The last part of the book, about 60 pages, is devoted to the Laplace transform, with applications to linear differential equations with constant coefficients, difference equations, and the partial differential equation of the vibrating string. The latter equation is also emphasized in the chapter on Fourier series.

The book is accurately and clearly written, and is a welcome text in Advanced Calculus.

University of Texas H. S. Wall

College Algebra. By M. Richardson. Prentice-Hall, Inc., New York, 1947. xvi + 472 pages.

In the Preface for the Instructor at the beginning of this book there are found the following sentences:

1) *"Stress is laid upon explanation of fundamental concepts and reasoning which the mathematics and science major is often over-optimistically expected to absorb by osmosis."*

2) "For a class which is going to omit a certain proof, it will be no more difficult to omit a correct proof than an inadequate one; but to teach incorrect reasoning as though it were correct can only undermine the student's future progress in and understanding of mathematics."

Thus the keynote of the book is struck at once; and it is seen that the author feels a discussion need not be humorless even though it appears in a mathematics text. This becomes more delightfully obvious as one proceeds.

Interesting historical notes appear frequently. There is an occasional reference to an unsolved problem, like Goldbach's conjecture that each even integer greater than 2 is the sum of two primes. Here and there throughout the book are found starred sections and starred exercises, more difficult than the rest of the material, which may be omitted if the instructor desires. The author gives in footnotes references to texts in which more advanced treatment of various topics can be found. Answers to odd-numbered problems appear at the end of the book, together with ten pages of convenient tables and a six-page index.

Something unfortunate seems to have happened between pages 117 and 120, 125 and 128, 133 and 136, 137 and 140. In each instance two blank pages intervene, which should obviously contain a connecting link between the page preceding and the page following them. The present reviewer dislikes the author's method of verifying an identity. She prefers to change one side until it takes the form of the other, but hesitates to emphasize the matter since it is one on which nearly every mathematics teacher can easily be roused to combat.

Anyone who wants to teach a course in college algebra can find in this book enough material to satisfy him. Physical scientists will note with pleasure applications to physics and astronomy, together with an explanation of the "terminology of variation" and of the "use of powers of ten in scientific writing."

Wellesley College. Marion E. Stark

PROBLEMS AND QUESTIONS

Edited by

C. G. Jaeger, H. J. Hamilton and Elmer Tolsted

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California. Contributions must be typed and figures drawn in india ink.

SOLUTIONS

No. 17. Proposed by *Pedro A. Piza,* San Juan, P. R.

Let x, y, and z be any three arbitrary non-zero integers, and let p be a prime number greater than 2. Prove that for any value of p whatsoever

$$[(x+y)(x+z)]^p - [x(x+y+z)]^p - [yz]^p = pxyz(x+y)(x+z)(x+y+z)k$$

where k is always an integer.

Solution by *C. W. Trigg,* Los Angeles City College.

In the general coefficient $p(p-1)\cdots(p-k)/1\times 2\cdots(k+1)$ of the expansion $(a+b)^p$, $(k+1)<p$ except in the $(p+1)$th term. All the binomial coefficients are integers, so if p is prime each coefficient (other than the first and last ones) is an integer multiple of p. Then $[(x+y)(x+z)]^p = [x(x+y+z) + yz]^p = [x(x+y+z)]^p + px(x+y+z)yz[f(x,y,z)] + [yz]^p$, where $f(x,y,z)$ is an integer. So, in the proposed equality the left hand member N is an integer multiple of $pxyz(x+y+z)$.

If p is odd, $[x(x+y+z)]^p = [(x+y)(x+z) - yz]^p = (x+y)(x+z)[g(x,y,z)] - [yz]^p$, where $g(x,y,z)$ is an integer. It follows that N is an integer multiple of $(x+y)(x+z)$.

Therefore, if p is an odd prime, $N = pxyz(x+y)(x+z)(x+y+z)k$, where k is an integer.

No. 22. Proposed by *Pedro A. Piza,* San Juan, Puerto Rico.

Let x and n be any two positive integers and let $\Sigma^n x^2$ stand for the nth iterated summation of all the squares from 1 to x^2 inclusive. For instance, $\Sigma 4^2 = 30$, $\Sigma^2 4^2 = 50$, (that is, the sum of the sums of all the squares from 1 to 16 inclusive), and $\Sigma^5 4^2 = 156$ (that is, the sum of the sums of the sums of the sums of the sums of all the squares from 1 to

16 inclusive). Prove that in general

$$\Sigma^n x^2 = x(x+1)(x+2)(x+3) \cdots (x+n)(2x+n)/(n+2)!$$

Solution by *C. W. Trigg,* Los Angeles City College.

It is well-known that $\Sigma^1 x^2 = x(x+1)(2x+1)/3! = (2x^3+3x^2+x)/3!$ It follows that $\Sigma^2 x^2 = (2\Sigma x^3 + 3\Sigma x^2 + \Sigma x)/3! = \frac{1}{3!}\left[\frac{2x^2(x+1)^2}{4} + \frac{3x(x+1)(2x+1)}{6} + \frac{x(x+1)}{2}\right] = x(x+1)(x+2)(2x+2)/4!$

We define $x^{(k+1)} = x(x+1) \cdots (x+k)$, Then by a well-known formula in finite integration $\Sigma x^{(k+1)} = x^{(k+2)}/(k+2)$. Now if $S = \Sigma^k x^2 = x(x+1)(x+2) \cdots (x+k)(2x+k)/(k+2)! = \frac{2x(x+1)(x+2)\cdots(x+k)(x+k+1)}{(k+2)!} - \frac{x(x+1)(x+2)\cdots(x+k)}{(k+1)!}$

then $\Sigma^{k+1} x^2 = \Sigma S = \frac{2x(x+1)\cdots(x+k+1)(x+k+2)}{(k+2)!\,(k+3)} - \frac{x(x+1)(x+2)\cdots(x+k+1)}{(k+1)!\,(k+2)} = \frac{x(x+1)\cdots(x+k+1)}{(k+3)!}[2(x+k+2) - (k+3)] = x(x+1)\cdots(x+k+1)(2x+k+1)/(k+3)!$ Since the given relation holds for $n = k+1$ if it holds for $n = k$, and it does hold for $n = 1$ and $n = 2$, then it holds for all positive integer values of n.

PROPOSALS

27. Proposed by *C. W. Trigg,* Los Angeles City College.

In a square of side "a" a right triangle is inscribed with one acute angle at a corner of the square and the other two vertices on the sides of the square non-adjacent to that corner. If one of the legs of the triangle is equal to the other leg plus the side of the square, find the longer leg of the triangle.

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28. Proposed by *W. R. Talbot,* Jefferson City, Missouri.

Find the form of the roots of a cubic function for which the abscissas of the roots, bend points, and inflexion point are distinct integers.

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29. Proposed by *Norman Anning,* Ann Arbor, Michigan.

Given that n is any positive integer greater than 1, show that the curve $\dfrac{1+y}{1-y} = \left(\dfrac{1+x}{1-x}\right)^n$ has three and only three points of inflection.

30. Proposed by *Victor Thébault*, Tennie, Sarthe, France.

Given a circle (O), of center O, tangent to two rays Ax and Ay, a variable tangent meets Ax at B and Ay at C. Show that each of two sides of the triangle which has the orthocenters of triangles AOB, BOC, COA as vertices pass through fixed points and that the third side has a conic as its envelope.

31. Proposed by *Victor Thébault*, Tennie, Sarthe, France.

Through a point P inside the face ABC of a tetrahedron $ABCD$ draw parallels to the edges DA, DB, DC which meet the planes of the faces BCD, CDA, DAB in A_1, B_1, C_1, respectively. Determine the position of P so that the volume of the tetrahedron $PA_1B_1C_1$ will be a maximum.

32. Proposed by *Victor Thébault*, Tennie, Sarthe, France.

Find all four-digit numbers *abcd* and *aecd* which are perfect squares.

33. Proposed by *F. M. Steadman*, Los Angeles, California.

Let R denote the proportion of the upper celestial hemisphere obscured at a point P by a horizontal circular area of radius a whose center is vertically above P at distance x. Express x in terms of a and R. In particular, find x in centimeters when the diameter of the circular area is one meter for $R = 1/2,\ 1/4,\ 1/8$.

The usual measure of solid angles is entirely analogous to the radian measure of plane angles. Is there a possibly useful measure of solid angles which is analogous to degree measure of plane angles and applicable to measurement of lens stops in photography?

MATHEMATICAL MISCELLANY

Edited by

Marian E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: MARIAN E. STARK, Wellesley College, Wellesley 81, Mass.

A Queer Set of Equations

From the following set of five equations in the five unknowns, $v, w, x, y, z,$ we can obtain for the consecutive unknowns: one value, the value one, any value, a zero value, and no value, respectively, as we show below.

$$
\begin{array}{rrrrrrrl}
3v & & -\,4x & +\;\;y & & +\,2vx & & = 6\\
 & 4w & & +\,2y & -\,3z & & +\,3wz & = 7\\
v & & -\,2x & +\,2y & & +\;\;vx & & = 2\\
 & 2w & & +\;\;y & -\,2z & & +\,2wz & = 4\\
 & w & & & -\;\;z & & +\;\;wz & = 2
\end{array}
$$

We derive the results by multiplying the equations by certain numbers and adding the results: These multipliers are given in the order of the equations they multiply.

Using the multipliers 1, 0, -2, 3, -6, we get $v = 2$, which is one value, although it is two.

Using 0, 1, 0, -2, 1, we get $w = 1$, the value one.

Substituting $v = 2$ in the equations, we find that they do not contain x at all, so x can have any value.

Using the multipliers 0, 0, 0, 1, -2, we get $y = 0$.

Using 0, -1, 0, 2, 0, we get $wz - z$ 1, but since $w = 1$, this gives $0 = 1$ whatever the value of z, so z can have no value.

Tufts College — William R. Ransom

Minimum Property of the Symmedian Point

A field in mathematics gains in esthetic appeal not only from the elegance of the details which are presented but also from the coherence of the whole treatment. Perhaps the most impressive proposition presented in a course on modern synthetic geometry is one which is not even stated explicitly, namely that the obviously pertinent properties of the triangle and circle can be treated elegantly and adequately by the methods of pure geometry. Perhaps no writer would care to defend this conjecture, but we may regard it as an esthetic gain when we remove an exception to the above generalization.

Specifically, the object of this note is to give a synthetic proof of the proposition that the point within the triangle such that the sum of the squares of its distances from the sides of the triangle be a minimum is the symmedian point. One may infer from P. H. Daus, College Geometry, p. 101 and R. A. Johnson, Modern Geometry, p. 216 that the synthetic approach is not generally known.

If this proposition is to be mentioned at all, it seems entirely appropriate to state and prove the proposition that the point within a triangle such that the sum of the squares of its distances from the vertices be a minimum is the centroid. Several proofs are possible. Johnson uses the law of cosines. A similar very neat proof can be given using vectors and dot products. A very simple proof can be based on the consideration that the locus of a point moving so the sum of the squares of its distances from B and C be constant is a circle centered at the midpoint A' of BC. Since the point is closest A when it is on the median AA', the minimum property of the centroid is immediate.

Now the minimum property of the symmedian point is very easily established, for if an arbitrary point P within a triangle is not the centroid of its own pedal triangle, it cannot have the required minimum property, for the distances from this centroid K to the sides of the given triangle are not greater than the distances to the vertices of the pedal triangle. Thus only if P coincides with K will it have the property that the sum of the squares of its distances from the sides of a triangle be a minimum, and then (P. H. Daus, College Geometry, p. 102) it is the symmedian point.

Arizona State College J. H. Butchart

Indicative of the wide interest in mathematics and perhaps in the type of magazine that the Mathematics Magazine represents is the fact that this magazine already goes into the following twenty eight countries outside continental U. S.:

Argentina	England	Philippine Is.
Australia	France	Poland
Belgium	Germany	Puerta Rica
Brazil	Greece	Spain
Canada	India	Suede
China	Ireland	Switzerland
Columbia	Malaya	Syria
Denmark	Mexico	U.S.S.R.
Dominican Re.	Nova Scotia	Venezuela

West Africa

Consideration of this list and similar lists of subscribers to other mathematical magazines probably had some influence on our forming the decision to make replacements on our editorial board on an international basis. A meightier influence was the fact that mathematics is a universal language, one of the many cultures that all peoples have in common emphasis upon which is good for these common cultures and for international relations. Several invitations have been sent out. To date we are happy to report acceptances by Prof. René Maurice Fréchet of the University of Paris, Prof. Nilos Sakellariou of the University of Athens and Prof. N. E. Norland of the University of Copenhagen.

Glenn James

The Institute fcr Numerical Analysis of the National Bureau of Standards, located at the University of California, Los Angeles, offers a number of research fellowships during the summer of 1949, and the academic year 1949 - 1950, to qualified graduate students in mathematics and mathematical physics. Fellows must be enrolled in an accredited college or university. Research work performed at the Institute may be applied toward a thesis for an advanced academic degree.

Fellows will work at the Institute and will be expected to perform mathematical research aimed at methods for advancing the applications of high speed automatic digital computing machinery. Individual work schedules may be arranged. Stipends will be based on full-time annual salaries of $2,294 for master's degree candidates, and $3,727 for doctoral candidates.

Inquiries and requests for application forms should be addressed to the Chief, Institute for Numerical Analyisis, 405 Hilgard Avenue, Los Angeles 24, California.

Here is this month's letter from one of the *Magazine's* readers.

"It gives me considerable satisfaction to enclose the accompanying check renewing my sponsoring subscription to the M.M. The past issues have been excellent, and of a type which justify the publication's place in the mathematics field.

"I have one minor criticism. The issues have not been of uniform page size.-----

"Please accept my appreciation of the very fine work you have done in relation to the magazine.

Sincerely,
Alan Wayne"
Flushing, L. I.

(Received by Professor Glenn James, with permission to quote.)

News About French Mathematics, from Maurice Fréchet.

Grave difficulties have been encountered by printers in France: lack of paper, then of coal and light, and finally of good workers (those idle during the war having found their way into other trades). Thus prices have continually increased and salaries being behind, sales are more difficult.

Gauthier - Villars did not try to restore everything at first and so issued a few books in trying to cope first with the periodicals. This has been practically done. The most delayed periodical has been Annales d l'Institut Henri Poincaré. Its IXth volume appeared in 1939. However, the Xth volume is now in course of publication — "fascicule 1" (Pollacek) appeared in 1946, fascicule 2 (Van Vleck), 3 (R. A. Fischer) followed, while 4 (Fréchet) is in course of printing and a number of other papers are in stock to follow. In 1947, Memorial Des Sciences Mathematiques published the pamphlets 105 (Humber et Collombo), and 106 (Bergmann) while 107 is in the press. The other mathematical periodicals at Gauthier — Villars are now appearing at correct time: Comptes Rendus de l'Académie des Sciences, each week; Journal de Math., each 3 months; Annales Ecole Normale Supericure, 2 a year; Bulletin Societé Math. France, one volume numbered I to IV in one copy provisionally each year; Bull. Sciences Math., fasc. Jan. - Feb. 1948 has appeared.

There is still a delay to cope with concerning books. "Processus Stochastiques et Mouvement Brownien" by Paul Lévy has appeared (October 1948). The second edition of Fréchet's first volume of "Recherches Theoriques Modernes Sur Le Calcul de Probabilités" was prepared and has been awaiting publication since August 1945 - it has been revised a second time and is still awaiting publication. This is only one instance of delay in printing.

Eight Days of Fun and Mathematics — Announcing — An Institute for Teachers of Mathematics in New England - to be held on the Wellesley College Campus, Wellesley, Mass. August 23 through 30, 1949.

The program will include speakers on the latest developments in pure mathematics and others on applications of mathematics by leading mathematicians from college faculties, business, research, government agencies and industry. There will also be discussion groups on methods of teaching; and an organized program of trips and other recreations. A real mixture of ideas and entertainment!

Watch this magazine for details or write to:

Ralph F. Ward
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Brookline, Mass.

or

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OUR CONTRIBUTORS

M. Zbigniew Krzywoblocki was born in 1904 in Lwow, Poland. He attended Lwow Institute of Technology (Diploma Ingenieur, '36), Warsaw University (Mathematics Department, '38—'39), Brooklyn Polytechnic Institute (M. Ae. E., '43, Ph.D., '44), Brown University (M.S. in applied Mathematics, '45), and Stanford University (M.A. in Pure Mathematics, '46). He was an instructor in the Mathematics Department (Chair of Projective Geometry) and in the Aerodynamics Chair of the Lwow Institute of Technology. Later he was Research Associate, Lwow Institute of Aeronautics and a visiting lecturer, Lwow Institute of Technology, Teaching Associate at Brooklyn Polytechnic and Fellow, Program of Advanced Mechanics, Brown University. His present position is that of Associate Professor of Theoretical Aerodynamics and Gas-Dynamics at the University of Illinois. He is an Associate fellow of the Royal Aeronautical Society, Associate Fellow of the Institute of Aeronautical Sciences, Member of the American Mathematical Society, Member of Sigma Xi.

Duane M. Studley, Director, Foundation Research, Colorado Springs, was born in Chicago, Illinois in 1916. He attended public schools in Colorado Springs, Colorado and Colorado College (Assoc. in Arts, '38). Mr. Studley is a member of the American Mathematical Society, the Institute of Mathematical Statistics, the Association for Symbolic logic, the American Association for the Advancement of Science and the American Physical Society. Besides holding the position referred to above, he is Director of Popular Education Distribution. His various interests include physics and logic as well as mathematics.

Bancroft H. Brown, Professor of Mathematics at Dartmouth College, was born in Hyde Park, Massachusetts in 1894. He attended Brown University (A.B. '16, A.M. '17) and then completed his graduate work at Harvard (Ph.D. '22). From 1919—21 Dr. Brown was an instructor at Harvard, and then joined the faculty at Dartmouth, where he became Assistant Professor in 1924 and Professor in 1931. He is a member of the A.A.A.S., Am. Math. Soc., Math. Assn. and the New Hampshire Academy of Sciences, and was president of the latter organization in 1940. His particular mathematical interest is geometry.

A biographical sketch of Estelle Mazziotta has appeared in a recent issue of this magazine, and that of M. Woodbridge will be published in the next issue.